Generalised turbulence spectra for broadband noise predictions with the Random Particle Mesh method

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Abstract

For better comparison to fan broadband noise experiments the Random Particle Mesh (RPM) method is extended to generalised turbulence spectra. The RPM method synthesises turbulent fluctuations by filtering white noise with a Gaussian filter kernel, which in turn gives a Gaussian spectrum. The Gaussian function is smooth and its derivatives and antiderivatives are again Gaussian functions; the Gaussian filter is efficient and finds wide-spread applications in stochastic signal processing. However in many applications Gaussian spectra are not matching physical spectra. E.g. in turbo-machines, the von Kärmán, Liepmann, and Modified von Kármán spectra are the most relevant model spectra. In the current paper we show how to analytically derive weighting functions to realise arbitrary spectra which are isotropic and solenoidal using a superposition of weighted Gaussian spectra of differing length scales. The analytic weighting functions for the von Kármán, the Liepmann, and the Modified von Kármán spectra are derived subsequently. Finally it is shown how to discretise the weighting functions to use a limited number of Gaussian spectra only. The effectivity of the discrete method is demonstrated by realising a von Kármán velocity spectrum using the RPM method.

Keywords:
Synthetic Turbulence, Isotropic Turbulence, Broadband Noise Modelling, Gaussian filter, Gaussian transformation, Gaussian spectrum, von Karman spectrum, Liepmann spectrum, Arbitrary spectra, Fast Random-Particle-Mesh Method

1. Introduction

To generate a stochastic noise signal of a certain spectral shape it is possible to convolute a white noise signal by a filter kernel of an appropriate shape \cite{1}.

One of the most common filter kernels is the Gauss filter kernel that realizes a Gaussian spectrum. The Gaussian filter is very simple and time efficient as it has beneficial characteristics: Its derivatives and antiderivatives are again Gaussian shape; The filtering decouples in multi-dimensional space and fast filter methods are available, such as Purser \cite{2} and Young & Van-Vliet \cite{3} filters.

But Gaussian spectra seldom represent the physics of turbulence. Here more elaborate spectra are needed, such as Kolmogorov, von Kármán or Liepmann spectra. For these spectra the filter kernels are very complicated and fully coupled in space, as shown by Dieste and Gabard \cite{4}.

Siefert et al. \cite{5} used a superposition of empirically weighted Gaussian spectra of different length scales to realize the Kolmogorov spectrum. The subsequently derived method has been applied to synthesise von Kármán spectra, successfully benchmarking the RPM method for leading edge noise by the authors \cite{6}, and to synthesise Liepmann spectra for computing generic trailing-edge noise by Rautmann et al. \cite{7}.

The objective of the current communication is to determine an analytical weighting function by means of Gaussian transformation \cite{8}. The analytical weighting function is derived and stated for the von Kármán, the Liepmann and the Modified von Kármán spectra. Furthermore, an effective method is shown to discretise the weighting function to a limited number of Gaussian spectra. For this, rules are derived to the optimum number of used filters and their length scales. For validation, the realized velocity spectrum using the Random Particle Mesh (RPM) method \cite{11} is compared to the analytically derived velocity spectrum.

2. Method - Gaussian Transformation

At first the target model spectra are presented. Then the general derivation of the weighting functions are shown, followed by the application to the model spectra. The application to the RPM method is shown in three steps: At first the amplitude scaling is discussed, then weighting functions are discretised and finally its minimum resolution is briefly discussed. An example, generating a von Kármán spectrum by using the the discretised weighting function together with the RPM method, is presented.
2.1. The Spectra

2.1.1. The von Kármán Spectrum

The von Kármán Spectrum is close to measured spectra in turbomachinery. It satisfies the energy law distribution of \( k'^{4} = (k \lambda)^{4} \) for the large eddies which contain most of the energy and reproduces the -5/3 - law in the inertial subrange:

\[
E_{k}(\hat{k}) = \frac{55}{9\pi} u_{t}^{3} \lambda \left( 1 + k^{2} \right)^{3/2},
\]

with the mean turbulent velocity \( u_{t} \) defined by the turbulent intensity \( T_{u} \) and the mean flow velocity \( u_{0} \) as \( u_{t}^{2} = (T_{u} \cdot u_{0})^{2} \), the integral length scale \( \lambda \), the reduced wavenumber \( \hat{k} = k \lambda / k_{e} \) and \( k_{e} = \sqrt{\frac{\lambda^{5/3}}{\Gamma(1/3)}} \).

2.1.2. The Liepmann Spectrum

The Liepmann spectrum \([9]\) is defined as:

\[
E_{L}(k') = \frac{8 \pi u_{t}^{3} \lambda}{\pi} k'^{-4} \left( 1 + k'^{-2} \right)^{3},
\]

(2)

2.1.3. The modified von Kármán spectrum

According to Bechara \([10]\) the von Kármán spectrum can be modified to be representative over the entire wavenumber range including the dissipation subrange:

\[
E_{M}(\hat{k}) = E_{k}(\hat{k}) \exp \left(-2 \frac{k^{2}}{k_{e}^{2}}\right)
\]

with the Kolmogorov wavenumber \( k_{e} = \left( \frac{\epsilon}{\nu} \right)^{1/4} \), where \( \epsilon \) is the specific dissipation rate and \( \nu \) is the eddy viscosity.

2.2. Weighting function

According to Ewert et al. \([11]\) filtering of a white noise field with a Gaussian filter kernel of a specific length scale realizes a Gaussian spectrum of the form

\[
E_{G}(k) = \frac{4 \pi \lambda^{3}}{\pi^{3}} k^{-4} e^{-\sigma^{2}}.
\]

(4)

For convenience we normalise this such that its integral over all wavenumbers is one, i.e.

\[
\int_{0}^{\infty} e_{G}(k, \lambda) dk = 1 \Rightarrow E_{G}(k) = \frac{3}{2} u_{t}^{2} e_{G}(k, \lambda).
\]

(5)

We are looking for a weighting function \( f(l, \lambda) \) to realise an arbitrary spectrum \( e(k) \) of integral length scale \( \lambda \) as a superposition of Gaussian spectra \( e_{G} \) of length scales \( l \):

\[
e(k) = \int_{0}^{\infty} f(l, \lambda) e_{G}(k, l) dl = \int_{0}^{\infty} f(l) \frac{8 \pi}{3} \hat{k}^{4} \exp \left(-\frac{k^{2} l^{2}}{\pi} \right) dl.
\]

(6)

Note that only the weighting function \( f(l, \lambda) \) is dependent of the integral length scale \( \lambda \), for convenience we subsequently write \( f(l, \lambda) = f(l) \).

With the substitution \( l^{2} = \frac{\pi}{\sigma^{2}} \) and \( \frac{dl}{d\sigma} = -\frac{\sigma}{2 \sqrt{\pi \sigma^{2}}} \) we can write Equation (6) in a suitable manner for Gaussian transform as defined by Alecu et al. \([8]\):

\[
e(k) = \frac{1}{k_{e}^{4}} \int_{0}^{\infty} f\left( \frac{\pi}{2 \sigma^{2}} \right) \frac{\pi}{3 \sqrt{2} \pi^{3}} \exp \left(-\frac{k^{2} \lambda^{4}}{2 \sigma^{2}} \right) d\sigma^{2}.
\]

(7)

Hence, from the Gaussian transform the weighting function \( f(l) \) derives as

\[
f(l) = \sqrt{\frac{\pi}{2 \sigma^{2}}} \exp \left(-\frac{k^{2} \lambda^{4}}{2 \sigma^{2}} \right).
\]

(8)

The weighting function \( f(l) \) is proportional to the variance of a Gaussian filter with the length scale \( l \).

2.2.1. Von Kármán weighting function

With the von Kármán spectrum given in Eq. (1) the left-hand side of Equation (7) becomes

\[
p(\sigma) = \frac{110}{27\pi} \lambda^{5/3} \left( k_{e}^{5/3} \right) \exp \left(-\frac{1}{\left( k_{e}^{2} + k^{2} \lambda^{2} \right)^{3/2}} \right).
\]

(9)

The direct Gaussian Transform is given by Alecu et al. \([8]\) Eq.(4)):

\[
\mathcal{G} (p(\sigma)) = \frac{1}{\sigma^{2}} \sqrt{\frac{\pi}{2\sigma^{2}}} \left( \mathcal{L}^{-1}(p(\sqrt{s}))(t) \right)_{t = \frac{1}{\sigma^{2}}},
\]

(10)

where \( \mathcal{L}^{-1} \) is the inverse LAPLACE transform. Using the relation

\[
\mathcal{L}^{-1}\left( 1 \frac{1}{(p - \alpha)^{n}} \right)(t) = \frac{e^{\alpha t} t^{n-1}}{\Gamma(n)}.
\]

(11)

where \( \Gamma(n) = (n-1)! \) is the gamma function, we find for Eq. (2):

\[
\mathcal{G}(p(\sigma)) = \frac{55}{54 \sqrt{\pi} (17/6)^{5/3}} \frac{k_{e}^{5} \lambda^{2}}{2 \sigma^{2} \lambda^{2}} \exp \left(-\frac{k_{e}^{2} \lambda^{4}}{2 \lambda^{4} \sigma^{2}} \right).
\]

(12)

and the weighting function for the von Kármán spectrum is given by

\[
f_{k}(l) = \frac{1}{18 \Gamma(17/6)} \sqrt{\frac{k_{e}^{5} \lambda^{2}}{\pi^{2} l}} \exp \left(-\frac{k_{e}^{2} \lambda^{4}}{\pi^{2} l} \right).
\]

(13)

This function is plotted in Fig. 1

\[
\text{with } \frac{\lambda^{2}}{\gamma^{2}} = \frac{\sqrt{\lambda^{2}}}{2 \lambda^{2} \gamma^{2}} \text{ and } \gamma = \sigma^{2}
\]
2.2.2. LIEPMANN weighting function

With the LIEPMANN spectrum given in Eq. (2) the left-hand side of Equation (7) becomes

$$p(k) = \frac{16\Lambda^4}{3\pi} \left( \frac{1}{k^2} \right)^{3/2}$$

(14)

This is the form of the generalised Cauchy distribution shown in appendix of Ref. [8] with \( \nu = 2.5 \) and \( b = \frac{1}{\Lambda} \). We identify

$$p(x) = \frac{16\Lambda^4}{3\pi} \sqrt{\pi}(2.5)^2$$

(15)

The Gaussian transform of \( p^\nu(x) \) is given in [8] as

$$G^\nu(\sigma^2) = \frac{b^2/2}{\sigma^2 \sqrt{2\pi}\Gamma(2.5)} \left( \pi^{-1/2} \right) \left( \sigma^2 + b^2 \right)^{-\nu/2}$$

(16)

So the Gaussian transform of \( p(\sigma^2) \) is

$$G(\sigma^2) = \frac{2}{\sqrt{3\pi}\sigma^2} e^{-\frac{1}{6\pi\sigma^2}}$$

(17)

Finally, this yields the weighting function for a LIEPMANN spectrum as

$$f_L(l) = \frac{2}{\Lambda \pi} e^{-\frac{l^2}{\Lambda^2}}$$

(18)

This function is plotted in Fig. [1]

2.2.3. MODIFIED VON KÁRMÁN weighting function

With the modified von Kármán spectrum given in Eq. (3) the left-hand side of Equation (7) becomes

$$p(k) = \frac{E_M(k)/k^4}{\Lambda^3 k^{8/3}}$$

$$= \frac{110}{27\pi} \Lambda^3 k^{8/3} \left( \frac{1}{k^2 + \Lambda^2} \right)^{1/6} \exp \left( -\frac{2k^2}{k_d} \right)$$

(19)

The direct Gaussian Transform is again given by Eq. (10) and we write

$$G(p(\sigma)) = \frac{1}{\sigma^2} \sqrt{\pi} \frac{\Lambda}{2\sigma^2} \left( \frac{110}{27\pi} \frac{\Lambda^3 k^{8/3}}{k^2 + \Lambda^2} \right)^{1/6} \exp \left( -\frac{2k^2}{k_d} \right)$$

(20)

This can be solved analytically, with the partial fraction expansion [11, p.783]. The image function is defined by \( I(p) = H(p)/G(p) \), with \( G(p) \) being a polynome about \( p \). First, we determine the inverse Laplace function to \( H(p) \) and \( 1/G(p) \). We get the inverse Laplace function by applying the convolution theorem afterwards.

The inverse Laplace function of \( 1/G(p) \) is given by Eq. (11) as

$$L^{-1} \left( \frac{1}{1/G(p)} \right) = L^{-1} \left( \frac{1}{\left( \frac{1}{\Gamma(17/6)} \right)^{1/6}} \right)$$

(21)

and the inverse Laplace function of \( H(p) \) yields

$$L^{-1} \left( H(p) \right) = \frac{e^{(t-\frac{2}{\pi})^{1/6}}}{\Gamma(17/6)}$$

(22)

(23)

So using the convolution theorem

$$L^{-1} \left( H(p)/G(p) \right) = L^{-1} \left( \frac{1}{1/G(p)} \right) \ast L^{-1} \left( H(p) \right)$$

(24)

(25)

Equation (20) simplifies to

$$G(p(\sigma)) = \begin{cases} \frac{110\Lambda^{-2/3} k^{8/3}}{27 \sqrt{2\pi}} \left( \frac{1}{\sigma^2} \right)^{1/6} \exp \left( -\frac{k^2}{k_d} \right) & \text{if } \frac{1}{2\sigma^2} \geq \frac{k^2}{k_d} \\ \frac{55\pi}{18\pi} \Lambda^{2/3} \left( \frac{1}{\sigma^2} \right)^{1/6} \exp \left( -\frac{k^2}{k_d} \right) & \text{if } \frac{1}{2\sigma^2} \geq \frac{k^2}{k_d} \\ 0 & \text{else.} \end{cases}$$

(26)

and we find the weighting function for the modified von Kármán spectrum as

$$f_M(l) = \begin{cases} \frac{55\pi}{18\pi} \Lambda^{2/3} \left( \frac{1}{\sigma^2} \right)^{1/6} \exp \left( -\frac{k^2}{k_d} \right) & \text{if } l \geq \frac{\sqrt{2\pi}}{k_d} \\ 0 & \text{else.} \end{cases}$$

(27)

This function is plotted in Fig. [1] From the plot we see that the modified von Kármán and the von Kármán weighting functions
are identical in a region of length scales above the Kolmogorov scale. From the modified von Kármán weighting function a cut-off condition for the smallest relevant length scales can be derived as

\[ k_d \geq \frac{\sqrt{2\pi}}{l} \] (28)

2.3. Amplitude for fRPM realisation

As shown by Ewert [12], the fRPM method delivers in 2D a unsteady fluctuating streamfunction \( \psi \) and the turbulent velocity fluctuations \( v \) by convolution of a unity white-noise field \( \mathcal{W} \) with the Gaussian filter kernel \( G(t) \):

\[ v = \nabla \times \psi = \nabla \times \int \hat{A} G(x - x') \mathcal{W}(x, t) d^2x'. \] (29)

The filter kernel \( G \) is normalised such that the auto-correlation \( \psi(x, t)\psi(x, t) = 1 \) for \( \hat{A} = 1 \). For the model spectra the unity Gaussian energy spectra are scaled with the variance \( f(l) \) given in Equation (13), (18) or (27). In analogy to [12], the normalisation parameter \( k_i \) is included and the variance is taken to be \( \sigma_i^2 = \frac{4}{3} k_i \). Therefore the amplitude for each weighted realisation in 2D is given by

\[ \hat{A} = \sqrt{\frac{4k_i f(l)}{3\pi}}. \] (30)

Following the same argumentation it can be shown that in 3D space the amplitude, given in

\[ v = \nabla \times \psi = \nabla \times \int \hat{A} G(x - x') \mathcal{W}(x, t) d^3x', \] (31)

yields

\[ \hat{A} = \sqrt{\frac{2k_i f(l)}{3\pi}}. \] (32)

Note that the correlations of the velocity fluctuations given in Equations (29) for 2D and (31) for 3D perfectly match the complete correlation tensor of isotropic turbulence given in [12] Equation (9)]. Therefore the model spectra derived in this paper also realise isotropic and solenoidal turbulence as they are a superposition of Gaussian spectra.

3. Discretisation

The main idea of the weighting functions is to realise an arbitrary spectrum \( E(k) \) of length scale \( \Lambda \) with the superposition of a limited number, say \( 1 \leq n \leq N \), of Gaussian spectra \( E_i \) of length scales \( l_i \). For this we discretise the integral in Eq. (6) to

\[ E(k) = \int_0^\infty f(l)E_i(k, l)dl \] (33)

\[ \approx \sum_{n=1}^{N} f(l_n)E_i(k, l_n)\Delta l_n \] (34)

with the spacing \( \Delta l_n \).

For an efficient realisation we want \( N \) to be as small as possible. As many orders of the wavelength \( k \) have to be covered an exponential distribution of the length scales \( l_n \) seems natural. Analytical parameter variations show that a realisation with 5 filters per order of \( k \) variation is already sufficient. In Fig. 2a this case is shown: The green curve shows the analytical von Kármán spectrum with an integral length scale \( \Lambda \) and the black curve is the realisation with 10 weighted Gaussian spectra for two orders of frequency. The set of blue curves show the \( n = 10 \) Gaussian spectra of length scales \( l_i \) weighted with the analytical weighting function \( f_k(l_n) \) of Eq. (13) The pink curve in the middle of the set shows the Gaussian spectrum of the same integral length scale as the von Kármán spectrum, but with weighting already applied.

For exponential discretisation, we discretise \( l_n \) logarithmically by a number of \( N \) discrete points by \( l_i = 10^{a_i} \) with

\[ a_i = (i - 1) \frac{a_N - a_1}{N - 1} + a_1. \] (35)
To define the spacing $\Delta l_\alpha$, the trapezoidal rule is applied:

$$\Delta l_1 = (l_2 - l_1)/2$$

$$\Delta l_i = \frac{l_{i+1} + l_{i-1}}{2}$$

$$\Delta l_N = (l_N - l_{N-1})/2.$$

Note that the smallest used length scale $l_1$ is chosen to resolve the highest wavenumbers of interest, but the length scale does not have to be smaller than the Kolmogorov length scale $l_d = \frac{\nu}{\epsilon}$. For low wavenumbers no additional Gaussian spectra are needed as this region is efficiently covered if the largest length scale is $l_N = 4\Lambda$. This is due to the fact that all investigated model spectra, including the Gaussian spectrum, follow the power law $E(k) \sim k^2$ in the low wavenumber region.

4. Application

The method derived earlier has been successfully applied to reproduce the measured inflow turbulence spectrum used for the fundamental test 1 of fan benchmark workshop [6] with the fRPM method [1]. The energy spectrum discretisation shown in Fig. 2a is the same used for the benchmark. Turbulent fluctuations of an integral length scale of $\Lambda = 8$ mm had to be resolved in a frequency range of 100 $\text{Hz}$ to $10 \text{kHz}$. For a resolved von Kármán spectrum in that range, we need 10 Gaussian spectra ranging from length scales $l_{\text{min}} = \Lambda/5$ to $l_{\text{max}} = 4\Lambda$. In Fig. 2b, the resulting measured and synthesised turbulent spectra of the axial velocity component are compared; the results show close agreement with the target shape.

As each filtering process, generating the Gaussian spectra of length scales $l_\alpha$, is fully decoupled from the others, the method can be implemented thread-parallel. In this way the over-head for realising arbitrary spectra is easily handled by using additional computational power, resulting in negligible penalty in efficiency compared to a computation realising a Gaussian spectrum.

5. Conclusion

Analytical weighting functions are derived to realise the von Kármán, the Liepmann, and the modified von Kármán spectra of an integral length scale $\Lambda$ by integrating weighted Gaussian spectra over length scales $l$. This integral is discretised to suppress only a limited number of Gaussian spectra. If a logarithmic discretisation of the length scales $l$ is chosen, it is demonstrated that 5 Gaussian realisations per order of frequency are enough for a smooth resulting spectrum. The discretisation is limited in the lower frequency range by $l \leq 4\Lambda$ and the upper limit is either given by grid resolution or in case of the modified von Kármán spectrum the Kolmogorov length scale $l_d = \frac{\pi}{\epsilon}k_d$. The method has been validated by generating a von Kármán spectrum with the fRPM method by a superposition of 10 Gaussian realisations to resolve 2 orders of magnitude of the frequency range.

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References


