Truncated Calogero-Sutherland models

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A one-dimensional quantum many-body system consisting of particles confined in a harmonic potential and subject to finite-range two-body and three-body inverse-square interactions is introduced. The range of the interactions is set by truncation beyond a number of neighbors and can be tuned to interpolate between the Calogero-Sutherland model and a system with interactions among nearest and next-nearest neighbors discussed by Jain and Khare. The model also includes the Tonks-Girardeau gas describing impenetrable bosons as well as a novel extension with truncated interactions. All these systems are exactly solvable and exhibit a linear spectrum, with the effect of the interactions being absorbed in a nontrivial zero-point energy. We characterize the degeneracies and derive the canonical partition function. While the ground state wavefunction takes a truncated Bijl-Jastrow form, excited states are found in terms of multivariable symmetric polynomials. We numerically compute the density profile and one-body reduced density matrix of the ground state and discuss the effect of the strength and finite range of the interaction potential.

Quantum systems with inverse-square interactions play a prominent role across a wide variety of fields. They are ubiquitous in many-body physics where they have facilitated the understanding of fractional quantum Hall effect and generalized exclusion statistics [1–3]. Historically, their study played a key role in understanding the integrability of systems with long-range interactions and the development of asymptotic Bethe ansatz [4–6]. Following the pioneering works by Dyson [7] and Sutherland [4], their connection to random matrix theory has remained a fruitful line of research [8, 9]. They have also found applications in blackhole physics [10] and conformal field theory [11–16]. More recently, they have been explored in the context of quantum decay of many-particle unstable systems [17] and in the study of thermal machines in quantum thermodynamics [18, 19].

In the one-dimensional continuum space, a many-body system with inverse-square interactions is generally known as the Calogero-Sutherland model (CSM) [4, 20, 21]. The CSM occupies a privileged status among exactly-solvable models as a source of inspiration [22–24]. In its original form, it describes one-dimensional bosons with inverse-square interactions of strength λ that exhibit a universal Luttinger liquid behavior [11]. Under harmonic confinement, this interacting Bose gas is equivalent to an ideal gas of particles with generalized exclusion statistics [3]. It is then referred to as the rational Calogero gas. Its connection with random matrix theory is manifested in the ground-state probability density distribution, which takes the form of the joint probability density for the eigenvalues of the Gaussian β-ensembles with Dyson index β = 2λ [4, 9]. While the CSM has shed new light on interacting systems, it can be mapped to a set of noninteracting harmonic oscillators [25, 26]. Further, the CSM can be extended to account for fermionic statistics, internal degrees of freedom, and additional interactions, e.g., of Coulomb type [25, 26]. In particular, when the range of the interaction is truncated to nearest-neighbors, the system remains integrable when supplemented with a three-body term, as discussed by Jain and Khare [27].

In this work, we introduce a family of one-dimensional quantum systems with tunable inverse-square interactions that extend over a finite number of neighbors. These systems exhibit a linear spectrum and generally involve pair-wise interactions, that can be either repulsive or attractive, as well as three-body attractive interactions. The CSM and the Jain-Khare model are recovered as particular limits. Further instances include the Tonks-Girardeau gas, describing impenetrable bosons in one-dimension [28, 29], and its generalization to finite-range interactions with two- and three-body terms. We shall refer to all these systems as truncated Calogero-Sutherland models (TCSM). They constitute a rare instance among exactly solvable models as they involve interactions that can be tuned both as a function of the strength and range. After finding the many-body eigenstates, energy levels and their degeneracies we derive the canonical partition function in closed form. In addition, we numerically compute the ground state density profile and the one-body reduced density matrix of the ground state and discuss the role of the strength and finite range of the interactions in both local and nonlocal one-body correlation functions.

MODEL AND GROUND-STATE PROPERTIES

The ground state of the Calogero-Sutherland model [4, 20] is well known to be exactly described by the product of single- and two-particle correlations, i.e., a Bijl-Jastrow form [4, 20, 25, 26]. The same holds true for the model with inverse-square interactions between nearest neighbors and an attractive three-body term initially discussed by Jain and Khare [27, 30–32]. This observation prompts us to consider a ground state described by the many-body wavefunction

\[ \Psi_0(x) = C^{-1/2}_{N,\lambda,r} \phi(x) \phi(x), \]  

(1)

with

\[ \phi(x) = \exp \left[ -\frac{m \omega^2}{2\hbar} \sum_{i=1}^{N} x_i^2 \right], \]  

(2)

...
and
\[
\varphi(x) = \prod_{i<j, |i-j| \leq r} (x_i - x_j)\lambda. \tag{3}
\]

Here, \(x = (x_1, \ldots, x_N) \in \mathbb{R}^N\). We shall refer to the corresponding probability distribution as the finite-range Dyson model
\[
\Psi_0^2 = C_{N,\lambda,r}^{-1} \exp \left[ -\frac{m\omega}{\hbar} \sum_{i=1}^{N} x_i^2 \right] \prod_{i<j, |i-j| \leq r} (x_i - x_j)^{2\lambda}, \tag{4}
\]
as it reduces for \(r = 1\) to the short-range Dyson model \([27]\) and for \(r = N - 1\) to the Gaussian \(\beta = 2\lambda\) ensembles in random matrix theory for \(\lambda = 1/2, 1, 2\) \([8, 9]\).

For \(r < N - 1\), the wavefunction does not have full permutation symmetry upon exchange of any two coordinates. We therefore work within an ordered sector, \(\mathcal{O} = \{x \in \mathbb{R}^N | x_1 > x_2 > \cdots > x_N\}\). In this sector, the normalization constant is given by the multi-dimensional integral \(C_{N,\lambda,r} = \int dr^N x^\lambda \).

While the full permutation symmetry of the system is broken upon truncation of the interaction range, it can be restored by symmetrizing the Hamiltonian over all permutations of coordinates, as discussed in Ref. \([30]\) for the Jain-Khare model. The TCSM represents then a quantum fluid as opposed to a quantum chain of impenetrable particles. For instance, the corresponding ground state is obtained by
\[
\varphi(x) \rightarrow \frac{1}{\sqrt{N!}} \sum_{P \in S_N} \prod_{P(i)<P(j), |P(i)-P(j)| \leq r} (x_{P(i)} - x_{P(j)})^{2\lambda} \Theta_P(x), \tag{5}
\]
where \(P\) runs over the symmetric group \(S_N\) with \(N!\) permutations and the ordering of the sector is set by \(\Theta_P(x) = \{x \in \mathbb{R}^N | x_{P(1)} > x_{P(2)} > \cdots > x_{P(N)}\}\). For the sake of clarity, we shall focus on the fundamental sector \(\mathcal{O}\).

The ground state (1) is shown to be an eigenstate of the Hamiltonian
\[
\hat{\mathcal{H}} = \sum_{i=1}^{N} \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} m\omega^2 x_i^2 \right) + \sum_{i<j, |i-j| \leq r} \frac{\hbar^2 \lambda (\lambda - 1)}{m|x_i - x_j|^2} + \sum_{i<j<k, |i-j| \leq r} \frac{\hbar^2 \lambda^2}{m} \frac{r_{ij} \cdot r_{jk}}{r_{ij}^3 r_{jk}}, \tag{6}
\]
where \(r_{ij} = (x_i - x_j)e_x\) and \(e_x\) is a unit vector along the x axis. As a result, Hamiltonian (6) describes N particles harmonically confined in one dimension and interacting through a pairwise two-body potential as well as a three-body term. We note that the truncation of the interaction is not mediated by the relative distance between particles, but rather by the number of neighbors. As such, it represents a screening effect. The maximum range of the two-body interactions is \(r\) while that of the three-body interactions is \(2r\), as \(r < |i-k| \leq 2r\). The two-body interactions are attractive for \(\lambda \in (0, 1)\) and repulsive for \(\lambda \geq 1\). By contrast, the three-body interactions are always attractive over \(\mathcal{O}\). Indeed, the three-body term can be conveniently rewritten as
\[
\sum_{i<j<k, |i-j| \leq r} \frac{r_{ij} \cdot r_{jk}}{r_{ij}^3 r_{jk}} = - \sum_{i<j<k, |i-j| \leq r} \frac{1}{|x_i - x_j||x_j - x_k|}. \tag{7}
\]

For \(r = N - 1\) this term vanishes identically and one recovers the ground state of the Calogero-Sutherland model for indistinguishable bosons restricted to a sector \([4, 20]\). In particular, for \(r = N - 1\) and \(\lambda = 1\) one recovers the Tonks-Girardeau gas in a harmonic trap, describing one-dimensional bosons with hard-core interactions \([28, 29]\). As a result, we shall refer to the case with \(\lambda = 1\) and \(r < N - 1\) as a truncated Tonks Girardeau (TTG) gas. For \(\lambda = 0\) the system describes an ideal Bose gas. To have N distinguishable particles in the limit \(r = N - 1\), the inverse square interaction would need to be \(\lambda^2 - \lambda M_{ij}\), where \(M_{ij}\) permutes particle coordinates \([33]\). For \(r = 1\), the Hamiltonian (6) reduces to the model with nearest-neighbor interactions.

The ground-state energy of the system is given by
\[
E_0 = \frac{\hbar \omega}{2} (N + \lambda r(2N - r - 1)). \tag{8}
\]

This expression suggests a mean-field theory where the zero-point energy of N non-interacting particles is shifted by \(\hbar \omega\) times the number of interacting pairs of particles, \(r(2N - r - 1)/2\). Hence, the zero-energy contribution reproduces the well-known result for the full Calogero-Sutherland model when the range of the interactions is set to \(r = N - 1\). In this limit, the scaling is quadratic in the particle number N. Else, for \(r < N - 1\), \(E_0 \propto N \) and depends quadratically on the range of the interaction \(r\).

**EXCITED SPECTRUM AND CANONICAL PARTITION FUNCTION**

To determine the full spectrum of the Hamiltonian we use the ansatz \(\Psi = \Psi_0 \Phi\) for an arbitrary excited state. The time-independent Schrödinger equation for \(\hat{\mathcal{H}}\) \(\Psi = E\Psi\) becomes the eigenvalue equation
\[
(2\hbar \partial \hat{\mathcal{K}} - \hat{\mathcal{D}}_+) \Phi = \varepsilon \Phi, \tag{9}
\]
where \(\partial = \frac{m\omega}{\hbar}\), \(\varepsilon = \frac{2m\omega}{\hbar}(E - E^{(0)}),\) and
\[
\hat{\mathcal{K}} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}, \tag{10}
\]
\[
\hat{\mathcal{D}}_+ = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{|i-j| \leq r} \frac{2\lambda}{x_i - x_j} \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right), \tag{11}
\]
that satisfy
\[
[\hat{\mathcal{D}}_+, \hat{\mathcal{K}}] = 2\hat{\mathcal{D}}_+. \tag{12}
\]
Defining $\hat{K}' = -\frac{1}{2} \left( \hat{K} + \frac{E_0}{\hbar \omega} \right)$, $\hat{D}' = \frac{1}{2} \hat{D} +$, and $\hat{D}' = \frac{1}{2} \sum_i x_i^2$, these operators $\{\hat{K}', \hat{D}'_\pm\}$ are the generators of the SU(1,1) algebra,

$$[\hat{D}'_+, \hat{D}'_-] = -2\hat{K}', \quad [\hat{K}', \hat{D}'_\pm] = \pm \hat{D}'_\pm. \quad (13)$$

Under the action of the similarity transformation $\hat{A} = \Psi_0 \exp(-\hat{D}'_/4\omega)$, the spectrum of the TCSM Hamiltonian (6) is related to that of the Euler operator $\hat{K}$,

$$\hat{A}^{-1} (\hat{P}' - E_0, \hbar \omega, \lambda) \hat{A}^{-1} = \hbar \omega \hat{K}. \quad (14)$$

An eigenstate of $\hat{K}$ is a homogeneous symmetric monomial $S_n$ with eigenvalue equal to the degree $n$, $\hat{K} S_n = n S_n$. The n-th excited state of the TCSM can be found in an analogous way as discussed in Ref. [34] for the CSM model (the limit with $r = N - 1$), and is given by $\Psi_n = \Psi_0 \Phi_n$ with

$$\Phi_n = \exp(-\hat{D}'_/4\omega) S_n. \quad (15)$$

The TCSM Hamiltonian can be further simplified by the additional transformation,

$$\phi(x) \hat{A}_0 \hat{A}^{-1} (\hat{P}' - E_0, \hbar \omega, \lambda) \hat{A}_0^{-1} \phi^{-1}(x) =$$

$$-\frac{\hbar^2}{2m} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \frac{m \omega^2}{2} \sum_{i=1}^N x_i^2 - \frac{\hbar \omega}{2} N, \quad \phi(x) = \frac{\text{Gaussian function}}{\text{given by Eq. (2)}}. \quad (16)$$

where $\hat{A}_0 = \exp\left[-\hat{D}'_/\lambda = 0/4\omega\right]$ and $\phi(x)$ is the Gaussian function given by Eq. (2). Hamiltonian (6) can thus be mapped to that of N decoupled harmonic oscillators. For $r = N - 1$, it is well-known that the excitation spectrum and level degeneracy of the interacting system is equivalent to that of noninteracting bosons in a harmonic trap [25, 35]. For $r < N - 1$, the singular terms in the similarity transformation can lead to nonpolynomial solutions, and the degeneracy of each level can be difficult to determine [31, 32].

The full spectrum can alternatively be found using Calogero’s approach [20], which separates (6) into radial and angular parts and finds the solutions for each using the form,

$$\Psi(x) = \phi(x) \Phi_{n,k}(\rho^2) P_k(x), \quad (17)$$

where $\rho^2 = \sum_{i=1}^N x_i^2$ is the radial degree of freedom and $\phi(x)$ is given in Eq. (3). Here $P_k(x)$ is a symmetric homogeneous polynomial of degree $k$ that satisfies a generalized Laplace equation,

$$\hat{D}'_+ P_k(x) = 0. \quad (18)$$

The radial solution reads

$$\Phi_{n,k}(\rho^2) = \exp(-\bar{\omega} \rho^2 / 2) L_n^κ(\bar{\omega} \rho^2), \quad (19)$$

where $\bar{\omega} = \frac{E_0(\hbar \omega)}{\hbar \omega} + k - 1$ and $L_n^κ(\bar{\omega} \rho^2)$ is a generalized Laguerre polynomial

$$L_n^κ(\bar{\omega} \rho^2) = \sum_{m=0}^n \frac{(κ+n)}{(n-m)} (-1)^m \frac{(\bar{\omega} \rho^2)^m}{m!}, \quad (20)$$

with $n \in \mathbb{N}_0$. The generalized Laguerre polynomial as well as the coefficient $\nu$ can also be found from (15), by taking $S_<(2n + k) = \rho^{2n} P_k(x)$ and using the following relation [32],

$$\hat{D}'_+ \left[ \frac{n \hbar \omega}{2} \left( E_0, \hbar \omega, \lambda \right) + k - 1 \right] \rho^{2n} P_k(x). \quad (21)$$

The corresponding spectrum is linear,

$$\left(E - E_0, \hbar \omega, \lambda \right) = \hbar \omega (2n + k) = \hbar \nu \omega, \quad (22)$$

for $s \in \mathbb{N}_0$, as anticipated from (16) and is universal for variations of the CSM preserving SU(1,1) in one spatial dimension [20, 30, 31, 36, 37]. The advantage of this approach is that the level degeneracy is more transparent.

Here we explicitly calculate the first few $P_k(x)$ in the spectrum, though in principle the entire spectrum can be found using this method. Here we focus on the case for $\lambda > 0$, where the level degeneracy of the spectrum is different from that of the non-interacting Bose gas and the full CSM. We construct $P_k(x)$ as a linear combination of symmetric monomial functions [38],

$$P_k(x) = \sum_{\alpha} c_\alpha m_\alpha(x), \quad (23)$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ describes a partition where the parts $\alpha_i$ are positive integers listed in decreasing order $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_N \geq 0$. The monomial functions are given by the sum over all distinct permutations $S_\alpha$ of $x_\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$,

$$m_\alpha(x) = \sum_{\sigma \in S_\alpha} x_{\alpha_1}^{\sigma(1)} x_{\alpha_2}^{\sigma(2)} \cdots x_{\alpha_N}^{\sigma(N)}. \quad (24)$$

The weight of a partition $|\alpha| = k = \sum_{i=1}^N \alpha_i$ corresponds to the order of $P_k(x)$. Each $P_k(x)$ has a total of $M(k)$ different coefficients, where $M(k)$ is the multiplicity of distinct partitions with a given weight $|\alpha| = k$. The requirement for $P_k(x)$ to satisfy the Laplace equation (18) will place constraints on these coefficients.

When solving for the constraints, there are four different cases to consider that depend on $r, k$, and $N$. For $k \leq N$, the partition length $l(\alpha) \leq N$, $\forall \alpha$ with $|\alpha| = k$, and general expressions for the constraints on $c_\alpha$ can be found for all $r$. However, for $k > N$ the partition length $l(\alpha) > N$ for some $|\alpha| = k$. In this case, there is no general form for the constraints as they will depend on $N, r$ and $k$ and will not include $\{|c_\alpha|\}$ where the partition length $l(\alpha) > N$. We do not provide the constraints for $k > N$, but they can be found using the same approach as for $k \leq N$.

In table I, we list the constraints on the coefficients $c_\alpha$ in (23) for $r < N - 1$ and $k \leq N$. For $k \geq 3$, the level degeneracy is different in the case of $r < N - 1$ and $r = N - 1$. We first consider the former case.

For $r = 1$, these expressions reproduce the constraints presented in [30] for the Jain-Khare model. For $r < N - 1$, there are $N_k = M(k) - 1$ constraints on the coefficients of each $P_k(x)$. Consequently, $P_k(x)$ is a one-parameter family of solutions for each $k$ and $r < N - 1$. The normalization requirement
of each excited state yields only one distinct solution for \( P_k(x) \) for a given \( k \), and \( (n,k) \) are the corresponding quantum number that describe each quantum state.

The corresponding level degeneracy for each \( s = 2n + k \) reads

\[
\begin{align*}
    d(s) &= s + 1, \quad s = 0, 2, \ldots, 2p, \\
    d(s) &= \frac{s + 1}{2}, \quad s = 1, 3, \ldots, 2p - 1.
\end{align*}
\]

This is the same degeneracy structure as found for the 1D multispecies CSM \([37]\), which has an underlying \( SU(2) \) symmetry. The multispecies model generalizes the full CSM by allowing the masses \( m_i \) and interactions between particles \( \lambda_{ij} \) to vary. By absorbing the truncated range into the interaction strength so that \( \lambda \to \lambda_{ij} = \lambda \theta(r - |i - j|) \), \( \forall i, j \) and taking \( m_i = m, \forall i \), the mathematical construction of the multispecies model is related to the TCSM. However, there is a fundamental difference as a wavefunction of the multispecies CSM has compact support on a given sector, as the particles are indistinguishable. Different sectors correspond to different physical realizations of the ordering of the particles, that is preserved under the time evolution generated by the system Hamiltonian. By contrast, particles in the symmetrized TCSM are indistinguishable and different sectors are explored as a result of scattering among particles. As a result, the behavior of nonlocal correlation functions such as the one-body reduced density matrix or the momentum distribution differ in these two models. We shall discuss both local and nonlocal one-particle correlations of the TCSM in Section 6.

For \( r = N - 1 \), the number of constraints on \( P_k(x) \) decreases to \( \mathcal{N} = M(k - 2) \). Given \( k \), there are \( M(k) - M(k - 2) \) unique solutions for \( P_k(x) \). For example, table II shows the constraints for \( 3 \leq k \leq 5 \).

The corresponding level degeneracy for each \( s = 2n + k \) is \( M(s) \), as it satisfies

\[
\sum_{n,k} \left[ M(k) - M(k - 2) \right] \delta(2n + k - s) = M(s). \tag{26}
\]

and the excited states, \( L^\nu_\lambda \theta(p^2) P_k(x) \), can be expressed as a linear combination of the Hi-Jack Polynomials \([39]\). The level degeneracy directly relates to Calogero’s result \([20]\), where the \( N \) quantum numbers \( \{n_l\} \) are solutions to

\[
s = \sum_{l=1}^{N} n_l, \tag{27}
\]

which holds for all \( k \).
TABLE II. Excited states of the CSM in a sector. Constraints on coefficients $c_{\alpha}$ in (23) for the case $r = N - 1$ and $k \leq N$ with all-to-all pairwise interactions, when the three-body term vanishes.

<table>
<thead>
<tr>
<th>$k$</th>
<th>Constraints for $r = N - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = 3$</td>
<td>$6c_3 + 2c_{21}(N - 1) + 2\lambda(N - 1) \left[ 3c_3 - c_{21} + (N - 2)(c_{21} - \frac{c_{31}}{2}) \right] = 0$</td>
</tr>
<tr>
<td>$k = 4$</td>
<td>$12c_4 + 2(N - 1)c_{22} + 2\lambda(N - 1) \left[ 4c_4 - c_{31} + (N - 2)(c_{22} - \frac{c_{32}}{2}) \right] = 0$</td>
</tr>
<tr>
<td></td>
<td>$12c_{31} + 2(N - 2)c_{211} + 4\lambda \left[ 2c_4 - c_{22} + (3N - 5)c_{31} \right] + 2\lambda(N - 2) \left[ (N - 5)c_{211} - \frac{(N - 3)}{2}c_{1111} \right] = 0$</td>
</tr>
<tr>
<td>$k = 5$</td>
<td>$5c_5 + c_{32}(N - 1) + \lambda \left[ (5c_5 - c_{41})(N - 1) + (c_{32} - c_{3111}/2)(N - 2) \right] = 0$</td>
</tr>
<tr>
<td></td>
<td>$9c_{311} + (N - 3)c_{2111} + \lambda \left[ 3(4c_{31} - 2c_{221} + 2c_{311}) + (9c_{311} - 3c_{2111})(N - 3) + (c_{2111} - \frac{c_{2211}}{2})(N - 4)/(N - 3) \right] = 0$</td>
</tr>
<tr>
<td></td>
<td>$6c_{41} + 3c_{32} + c_{221}(N - 2) + \lambda \left[ (5c_5 + 3c_{341} - 2c_{32}) + (N - 2)(4c_{41} - c_{311} + 3c_{32} + (N - 4)c_{221} - \frac{c_{2211}}{2})(N - 3) \right] = 0$</td>
</tr>
</tbody>
</table>

By comparing these two cases, it is clear that the truncation not only significantly reduces the degeneracy level structure for $s > 2$, but also reduces the number of relevant quantum numbers from $N$ for the $r = N - 1$ case to only two quantum numbers ($n, k$) for $r < N - 1$.

Knowledge of the spectrum and the degeneracy of the energy levels allows one to compute the canonical partition function from which all equilibrium thermodynamics can be derived. For $r < N - 1$ the partition function reads,

$$Z_N^{(r)} = \sum_{n = 0}^{N} d(s) e^{-\beta E_n} = \frac{e^{-\beta E_{0,N,r}}}{(1 - e^{-2\beta \omega_0})(1 - e^{-\beta \omega_0})}$$

where $d(s)$ is given in Eq. (25). For $r = N - 1$ one needs to take into account the different level degeneracy and one recovers the partition function of the full CSM (see, e.g., [18, 19]).

**CORRELATIONS IN SYMMETRIZED MODEL**

The knowledge of the exact ground state of the Hamiltonian (6) allows us to investigate the role of the truncation of the interaction. To this end, we next focus on the characterization of one-body correlations. The fully symmetrized ground-state wavefunction reads,

$$\Psi_0^\uparrow(x) = \phi(x) \varphi^\uparrow(x),$$

where $\varphi^\uparrow(x)$ is given in Eq. (5). In particular, we consider the density profile of the ground state, defined as

$$n(x) = N \int_{\mathbb{R}^{N-1}} d^{N-1} x' |\Psi_0^\uparrow(x')|^2,$$

$$= \sum_{P_1, P_2' \in \sigma^p} \int_{\mathbb{R}^{N-1}} d^{N-1} x' \left| \Psi_0 \left( P_1(x) \right) \right|^2 \Theta_{P_1}(x),$$

where $x = (x, x_2, \cdots, x_N)$, $\sigma^p = \{(1 \cdots p), p = 1, \cdots, N\}$ is the set of $p$-cycles that effectively shifts $x$ to the right of each element in $x$, and the identity $\Theta_{P_1}(x) \Theta_{P_2'}(x) = \Theta_{P_1}(x) \delta_{P_2'}$ was used. For a fixed particle number $N$ and range $r$, the density profile develops signatures of spatial antibunching as the interaction strength $\lambda$ is increased. This is confirmed by the numerical integration using the Monte Carlo method in Figure 1(a) for $N = 4$, $r = 2$ for $\lambda = \{0, 1/2, 1, 2\}$. As $\lambda$ is increased, the density profile varies from a bell-shape function to a broader distribution that shows fringes and a number of peaks that equals the particle number $N$.

When $\lambda = 1$ and $r = N - 1$, even in the absence of symmetrization, the density profile reduces to that of a Tonks-Girardeau gas that describes one-dimensional bosons with hard-core interactions. In the ground state, its explicit form is $n(x) = \sum_{n = 0}^{N-1} |\phi_n(x)|^2$, where $\phi_n(x)$ is the single-particle eigenstates of the harmonic oscillator. This expression is identical to that of the density profile of polarized and spinless fermions [28, 29]. In the limit $r = 0$ one recovers the ideal Bose gas, with $n(x) = N|\phi_0(x)|^2$. The TCSM with $\lambda = 1$ is equivalent to a TTG gas and interpolates between these two limits for $0 < r < N - 1$. The visibility of the fringes in $n(x)$ diminishes then as the range of the interactions is decreased, see Fig. 1(b) for $N = 5$ and $r = \{1, 2, 4\}$.

We also consider the one-body reduced density matrix, defined as

$$\rho(x, x') = N \int_{\mathbb{R}^{N-1}} d^{N-1} x' \Psi_0^\uparrow(x) \Psi_0^\uparrow(x'),$$

$$= \sum_{P_1, P_2' \in \sigma^p} \int_{\mathbb{R}^{N-1}} d^{N-1} x' \left| \Psi_0 \left( P_1(x) \right) \right|^2 \Theta_{P_1}(x) \Theta_{P_2'}(x'),$$

where $x' = (x', x_2, \cdots, x_N)$ and the expression in the second line is suitable for Monte Carlo integration. Here, $\sigma^p$ is as previously defined. Figure 2 shows $\rho(x, x')$ for (a) $N = 5, \lambda = 1, r = 1$, (b) $N = 5, \lambda = 1, r = 2$, and (c) $N = 5, \lambda = 1, r = 3$. The maximum amplitude lies along the diagonal, and matches the density profile, e.g., $\rho(x, x) = n(x)$. The amplitude decreases when going away from the diagonal. This characterizes a loss of off-diagonal long range order, that decays much...
FIG. 1. **Density profile of the TCSM.** Particle density as a function of space for $N = 5$. (a) Increasing the interaction strength while keeping the range fixed ($r = 2$) leads to spatial antibunching reflected in the fringes of the density profile. (b) For a given strength of the interactions ($\lambda = 1$), the TCSM interpolates between the Jain-Khare model ($r = 1$) and the rational Calogero-Sutherland model ($r = N - 1$) as the range is increased, enhancing spatial antibunching. The case $\lambda = 1$ shown here can be thought of as a truncated Tonks-Girardeau (TTG) gas.

FIG. 2. **One-body reduced density matrix of the TCSM.** Plot of $\rho(x, x')$ as a function of space for $N = 5$ particles and interaction strength $\lambda = 1$, i.e., the TTG gas. The case $r = 1$ in (a) is contrasted with $r = 2$ in (b) and $r = 3$ in (b), showing that off-diagonal long-range order is suppressed as the interaction range $r$ increases. The color coding runs from dark to light with increasing value of $\rho(x, x')$.

faster for $r = 3$ than $r = 1$. This is consistent with the fact that interactions are suppressed as $r$ is decreased, approaching the ideal Bose gas for $r = 0$. The $r = 3$ system is closer to the Tonks-Girardeau gas describing impenetrable bosons in a harmonic trap, where the off-diagonal long range order has been shown to vanish in the thermodynamic limit [29].

**CONCLUSIONS AND OUTLOOK**

We have introduced a family of models describing one-dimensional particles confined in a harmonic potential and subject to inverse-square interactions among a finite number of neighbors. This family of truncated Calogero-Sutherland models involves pairwise interactions that can be repulsive or attractive and a three-body contribution that is always attractive. It interpolates between the Calogero-Sutherland model with full range interactions, when the three-body term vanishes, and the model introduced by Jain and Khare. The system can be understood as a quantum chain in the continuum space or a quantum fluid after restoring full permutation symmetry. In the latter case, it includes the Tonks-Girardeau gas that describes impenetrable bosons in one spatial dimension as well as a novel extension with truncated interactions. The TCSM remains exactly solvable in spite of the tunable strength and range of the interactions. We have shown that the spectrum is linear and derived the degeneracies of the energy levels to find the canonical partition function. By numerically computing the density profile and the one-body reduced density matrix we have demonstrated that increasing the interaction strength and range leads to spatial antibunching and suppresses off-diagonal long-range order.

This family of truncated models can be extended in a wide variety of ways to account for particles with internal structure [25, 26], multiple species [37], higher dimensions [40], anyonic and fermionic exchange statistics, and modified interactions and confining potentials [23]. One may envision as well an extension of our model to a variety of root systems.
with non-traditional reflection symmetries [41].

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