Largest Eigenvalue and Invertibility of Symmetric Matrix Signings

Charles Carlson Karthekeyan Chandrasekaran Hsien-Chih Chang Alexandra Kolla*

Abstract

The spectra of signed matrices have played a fundamental role in social sciences, graph theory and control theory. They have been key to understanding balance in social networks, to counting perfect matchings in bipartite graphs, and to analyzing robust stability of dynamic systems involving uncertainties. More recently, the results of Marcus *et al.* have shown that an efficient algorithm to find a signing of a given adjacency matrix that minimizes the largest eigenvalue could immediately lead to efficient construction of Ramanujan expanders.

Motivated by these applications, we investigate natural spectral properties of signed matrices and address the computational problems of identifying signings with these spectral properties. Our main results are: (a) NP-completeness of three problems: verifying whether a given matrix has a signing that is positive semi-definite/singular/has bounded eigenvalues, (b) a polynomial-time algorithm to verify whether a given matrix has a signing that is invertible, and (c) a polynomial-time algorithm to find a minimum increase in support of a given symmetric matrix so that it has an invertible signing. We use combinatorial and spectral techniques; our main new tool is a combinatorial characterization of matrices with invertible signings that might be of independent interest. We use our characterization and classic structural results from matching theory to find a minimum increase in the support in order to obtain invertible signings.

*Email: {ccarlsn2,karthe,hchang17,akolla}@illinois.edu

1 Introduction

The spectra of several graph-related matrices such as the adjacency and the Laplacian matrices have become fundamental tools in computer science research. They have had a tremendous impact in several areas including machine learning, data mining, web search and ranking, scientific computing, and computer vision. In this work, we investigate the spectrum and the invertibility of signings of matrices.

For a real symmetric $n \times n$ matrix M and a $n \times n$ matrix s taking values in $\{\pm 1\}$—which we refer to as a *signing*—we define the *signed matrix* $M(s)$ to be the matrix obtained by taking entry-wise products of M and s. We say that s is a *symmetric* signing if s is a symmetric matrix and an *off-diagonal* signing if s takes value $+1$ on the diagonal. Signed adjacency matrices (respectively, Laplacians) correspond to the case where M is an adjacency matrix (respectively, Laplacian) of a graph. Signed adjacency matrices were introduced as early as 1953 by Harary [21], to model social relations involving disliking, indifference, and liking. They have since been used in an array of network applications such as finding "balanced groups" in social networks where members of the same group like or dislike each other [21] and reaching consensus among a group of competing or cooperative agents [6]. Studies of spectral properties of general signed matrices as well as signed adjacency matrices have led to breakthrough advances in multiple areas such as algorithms [2, 8, 15, 16, 30, 34], graph theory [37, 41, 45, 49] and control theory [10, 27, 35, 38, 42].

In this work, we study natural spectral properties of symmetric signed matrices and address the computational problems of identifying signings with these spectral properties. The magnitude of the eigenvalues of symmetric signed adjacency matrices and Laplacians are fundamental to designing expanders [2, 8, 15, 16, 30] and in particular, Ramanujan expanders [34]. Motivated by this application, we examine symmetric signed matrices with bounds on the magnitude of the eigenvalues. We next examine the singularity and the invertibility of symmetric signed matrices. Both these properties are completely determined by the determinant. The determinant of signed matrices has a special significance in the computational problems of counting matchings and computing the permanent [37, 41, 45, 49]. In what follows, we elaborate on these significances, motivate the related computational problems and describe our results.

Spectra of signed matrices and expander graphs. The relation between the expansion of a graph and the second eigenvalue λ_2 of the adjacency matrix of the graph has been a celebrated fact for decades [5, 12]. The Alon-Boppana bound [36] states that $\lambda_2 \geq 2\sqrt{d-1} - o(1)$ for every d-regular graph. For this reason, graphs with $\lambda_2 \leq 2\sqrt{d-1}$ are optimal expanders and are called *Ramanujan*. The construction of Ramanujan graphs by Lubotzky, Philips, and Sarnak [33] was a landmark achievement, giving d-regular graphs with the best possible expansion. However, their construction only produced Ramanujan expanders of certain degrees. Efficient construction of Ramanujan expanders of *all* degrees remains an important open problem.

A combinatorial approach to this problem, initiated by Friedman [15], is to obtain larger Ramanujan graphs from smaller ones while preserving the degree. The smaller starting graph is known as the *base graph* while the larger graph is known as a *lift*. A 2-*lift* H of G is obtained as follows: Introduce two copies of each vertex u of G, say u_1 and u_2, as the vertices of H and for each edge $\{u, v\}$ in G, introduce either $\{u_1, v_2\}, \{u_2, v_1\}$ or $\{u_1, v_1\}, \{u_2, v_2\}$ as edges of H. There is a simple bijection between 2-lifts and symmetric signed adjacency matrices of G. Furthermore, the eigenvalues of the adjacency matrix of a 2-lift H are given by the union of the eigenvalues of the adjacency matrix of the base graph G (also called the "old" eigenvalues) and the signed adjacency matrix of G that corresponds to the 2-lift (the "new" eigenvalues). Marcus, Spielman, and Srivastava [34] showed that there is always a 2-lift of every d-regular bipartite graph whose new eigenvalues satisfy the Ramanujan bound, which by the Alon-Boppana bound is the best possible.

An important consequence of Marcus *et al.*'s result [34] is that there is an iterative procedure to obtain bipartite Ramanujan expanders of any degree d and any size: Start with a small d-regular Ramanujan expander (for example, $K_{d,d}$), find a symmetric signing that minimizes the largest eigenvalue of the signed adjacency matrix, and repeat. However, Marcus *et al.*'s result [34] is not constructive and their work raises the question of whether there is an efficient algorithm to find such a signing that minimizes the largest eigenvalue. Motivated by this application, we study the following decision version of the computational problem of finding a signing that minimizes the largest eigenvalue.

BoundedEvalueSigning: Given a real symmetric matrix M and a real number λ, verify if there exists an off-diagonal symmetric signing s such that the largest eigenvalue $\lambda_{\max}(M(s))$ is at most λ.

We note that Cohen [13], in a follow-up to the work of Marcus *et al.* [34], has shown an efficient algorithm to find bipartite Ramanujan *multi-graphs* of all degrees, via the method of *interlacing family*, without addressing the above-mentioned computational problem. It still remains open to efficiently construct bipartite Ramanujan *simple* graphs of all degrees. If BoundedEvalueSigning is solvable in polynomial time, then it immediately gives an efficient algorithm to construct simple bipartite Ramanujan graphs of all degrees. In our first result, we shed light on the question of finding a signing of a matrix in order to minimize the largest eigenvalue.

Theorem 1.1. BoundedEvalueSigning *is NP-complete.*

We remark that the hard instances generated by our proof of Theorem 1.1 are real symmetric matrices with non-zero diagonal entries and hence, it does not completely resolve the computational complexity of the problem of finding a signing of a given *adjacency matrix* that minimizes its largest eigenvalue. However, it gives some indication that the task of making the results by Marcus *et al.* [34] constructive would require techniques that are very specific to graphs and graph related matrices and cannot generalize to arbitrary matrices.

We recall that a matrix M is *positive semi-definite* if all its eigenvalues are non-negative. We show Theorem 1.1 by considering the following closely related problem:

PsdSigning: Given a real symmetric matrix M, verify if there exists a symmetric signing s such that $M(s)$ is positive semi-definite.

Theorem 1.2. PsdSigning *is NP-complete.*

Determinant of signed graphs. The determinant of signed adjacency matrices of graphs have been crucial to the study of several fundamental questions concerning graphs and linear systems [37, 41, 45, 49]. We mention some of these questions: *Pólya's scheme*: Given an adjacency matrix A, is there a signing of A such that the permanent of A equals the determinant of the signed matrix? *Sign solvability*: Given a real square matrix, is every real matrix with the same sign pattern (that is, the corresponding entries either have the same sign, or are both zero) invertible? *Pfaffian orientation*: Given a bipartite graph, does it have a *Pfaffian orientation*? *Even length circuits in digraphs*: Given a digraph, does it have no directed circuit of even length?

All of these questions are known to be equivalent to each other and in particular, closely related to the problem of counting the number of perfect matchings in a given bipartite graph (see for example, Thomas [45]).

The motivation for the above-mentioned questions largely arose from the fact that the complexity of computing the permanent of a matrix is fundamentally different from the complexity of computing the determinant [48]. Pólya [39] suggested the possibility of computing the permanent by somehow reducing the problem to computing the determinant of a related matrix: If A is a 0-1 square matrix, under what conditions is there a signed matrix B obtained from A such that the permanent of A equals the determinant of B? In a seminal work, Robertson, Seymour, and Thomas [41] gave a structural characterization of matrices that have this property (which they called *Pólya matrices*) and presented a polynomial time algorithm that decides whether a given matrix is a Pólya matrix.

In this work, we investigate questions concerning the invertibility of symmetric signed matrices. Our next result examines the problem of verifying whether *every* symmetric signing of a given matrix is invertible:

SingularSigning: Given a real symmetric matrix M, verify if there exists an off-diagonal symmetric signing s such that $M(s)$ is singular.

Theorem 1.3. SingularSigning *is NP-complete.*

We next consider the counterpart of SingularSigning, namely verifying whether *every* symmetric signing of a given matrix is singular:

InvertibleSigning: Given a real symmetric matrix M, verify if there exists a symmetric signing s such that $M(s)$ is invertible (i.e., non-singular).

In contrast to the SingularSigning problem, we show that the InvertibleSigning problem is solvable in polynomial time.

Theorem 1.4. *There exists a polynomial time algorithm to solve* InvertibleSigning*.*

Our algorithm for solving InvertibleSigning is based on a novel graph-theoretic characterization of symmetric matrices M for which every symmetric signing $M(s)$ is singular which we describe below. We believe that this characterization might be of independent interest. The *support graph* of a real symmetric $n \times n$ matrix M is an undirected graph G where the vertex set of G is $\{1, \ldots, n\}$, and the edge set of G is $\{\{u, v\} \mid M[u, v] \neq 0\}$. We note that G could have self-loops depending on the diagonal entries of M. For a subset S of vertices in graph G, let $N_G(S)$ be the *non-inclusive neighborhood* of S, that is, $\{u \in V \setminus S \mid \{u, v\}$ is an edge of G for some v in $S\}$. We recall that a subset S of vertices is said to be *independent* if there are no edges between any pair of vertices in S. Theorem 1.4 follows immediately from the following theorem.

Theorem 1.5. *Let M be a symmetric $n \times n$ matrix and let G be the support graph of M. The following are equivalent:*

1. *The signed matrix $M(s)$ is singular for every symmetric signing s.*
2. *There is a non-empty independent set Q such that $|N_G(Q)| < |Q|$, where none of the vertices in Q have self-loops.*

Moreover, there exists a polynomial-time algorithm to verify whether the signed matrix $M(s)$ is singular for every symmetric signing s.

A subset S of vertices is said to be *expanding* in G if $|N_G(S)| \geq |S|$. Thus, Theorem 1.5 gives a spectral characterization for the existence of a *non-expanding independent set* in a graph G: A graph G contains a non-expanding independent set if and only if every symmetric signed adjacency matrix of G is singular.

Graphs with expanding independent sets have been encountered frequently in the study of independent sets [4, 11, 46] as well as matchings [32, 47]. A *perfect* 2*-matching* in a graph G with edge set E is an assignment $x : E \to \{0, 1, 2\}$ of values to the edges such that $\sum_{e \in \delta(v)} x_e = 2$ holds for every vertex v in G (where $\delta(v)$ denotes the set of edges incident to v). We obtain the polynomial-time algorithm mentioned in Theorem 1.5 as well as Theorem 1.4 using a classic characterization by Tutte [47]. He showed that the existence of a non-expanding independent set is equivalent to the absence of a perfect 2-matching in the graph. Tutte's results also give a polynomial-time algorithm to verify the existence of a perfect 2-matching in a given graph (for example, see Lovász-Plummer [32, Corollary 6.1.5]).

Efficient algorithms for finding the solvability index of a signed matrix. The notion of *balance* of a symmetric signed matrix is crucial to social sciences and has been studied extensively [21, 23, 24, 29]. A signed adjacency matrix is *balanced* if there is a partition of the vertex set such that all edges within each part are positive, and all edges in between two parts are negative (one of the parts could be empty). A number of works [3, 20, 29, 44, 50, 51] have explored the problem of minimally modifying signed graphs (or signed adjacency matrices) to convert it into a balanced graph. Given a signed matrix A, the *frustration index* of A is the minimum number of non-zero off-diagonal entries of A whose deletion results in a balanced signed graph [1, 22]. A simple reduction from MaxCut shows that computing the frustration index of a signed graph is NP-hard [26].

In this paper, we introduce a related problem regarding signed matrices: Given a symmetric $n \times n$ matrix A, what is the smallest number of non-diagonal zero entries of A whose replacement by non-zeroes gives a symmetric matrix A' that has an invertible symmetric signing? Our main motivation is to study systems of linear equations in signed matrices that might be ill-defined, and thus do not have a (unique) solution and to minimally modify such matrices so that the resulting linear system becomes (uniquely) solvable.

For a real symmetric matrix M, the *solvability index* of M is the smallest number of non-diagonal zero entries that need to be converted to non-zeroes so that the resulting *symmetric* matrix has an invertible symmetric signing. We emphasize that the support-increase operation that we consider preserves symmetry, that is, if we replace the zero entry $A[i, j]$ by $A'[i, j] = \alpha$, then the zero entry $A[j, i]$ is also replaced by $A'[j, i] = \alpha$. We give an efficient algorithm to find the solvability index.

Theorem 1.6. *There exists a polynomial time algorithm to find the solvability index of a given real symmetric matrix.*

In order to show Theorem 1.6, we exploit the combinatorial characterization given in Theorem 1.5 and the results of Tutte mentioned above to reduce the problem of computing solvability index to the problem of adding minimum number of edges to a given graph so that the resulting graph contains a perfect 2-matching. We solve the resulting minimum edge addition problem using the Gallai-Edmonds decomposition [14, 17, 18] and by exploiting its connection to the additive integrality gap of the fractional matching linear program.

Remark. Our techniques to find the solvability index can also be used to find the largest principal submatrix that has an invertible symmetric signing. We defer the result to the full-version of the paper.

Since all of our results are for symmetric signings, we will just use the term *signing* to refer to a symmetric signing in the rest of the paper.

1.1 Paper Organization

The paper is organized as follows. In Section 2 we prove the NP-completeness results, namely Theorems 1.1, 1.2, and 1.3. The proofs of Theorems 1.2 and 1.3 use similar tools, so it is convenient to prove them together. The proof of Theorem 1.1 follows as a corollary of the NP-completeness of a variation of the PsdSigning problem. In Section 3 we show our combinatorial characterization of matrices with invertible signings and provide a polynomial-time algorithm to verify whether a given matrix has an invertible signing, thereby proving Theorem 1.5. In Section 4, we use the characterization from Section 3 and structural results from matching theory to give a polynomial-time algorithm to compute the solvability index, and thus prove Theorem 1.6.

2 Hardness of Signing Problems

In this section we prove Theorems 1.1, 1.2, and 1.3.

We use the notion of *Schur complement* in our reduction. The following lemma summarizes the definition and the relevant properties of the Schur complement.

Lemma 2.1 (Horn and Johnson [25])**.** *Let D be a symmetric matrix whose blocks are of the following form (with appropriate dimensions):*

$$D = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}.$$

Suppose A is invertible. Then the Schur complement *of C in matrix D is defined to be $D_C := C - BA^{-1}B^T$. We have the following properties:*

(i) Suppose A is positive definite. Then, D is positive semi-definite if and only if the Schur complement of C in D, namely D_C, is positive semi-definite.

(ii) $\det(D) = \det(A) \cdot \det(D_C)$.

In order to show the NP-completeness results, we reduce from the Partition problem, which is a well-known NP-complete problem [28]. We recall the problem below:

Partition: Given an n-dimensional vector b of non-negative integers, determine if there is a ± 1-signing vector z such that the inner product $\langle b, z \rangle$ equals zero.

We have the ingredients needed to prove Theorems 1.2 and 1.3.

Proof of Theorems 1.2 and 1.3. Both PsdSigning and SingularSigning are in NP since if there is an (off-diagonal) signing of the given matrix that is positive semi-definite or singular, then this signing gives the witness. In particular, we can verify if a given (off-diagonal) symmetric signed matrix is positive semi-definite or singular in polynomial time by computing its spectrum [19].

We show NP-hardness of PsdSigning and SingularSigning by reducing from Partition. Let the n-dimensional vector $b := (b_1, \ldots, b_n)^T$ be the input to the Partition problem, where each

b_i is a non-negative integer. We construct a matrix M as an instance of PsdSigning/SingularSigning as follows: Consider the following $(n+2) \times (n+2)$-matrix

$$M := \begin{bmatrix} I_n & b & \mathbf{1}_n \\ b^T & \langle b, b \rangle & 0 \\ \mathbf{1}_n^T & 0 & n \end{bmatrix},$$

where I_n is the $n \times n$ identity matrix and $\mathbf{1}_n$ is the n-dimensional column vector of all ones. Claims 2.2 and 2.3 prove the correctness of the reduction to PsdSigning and SingularSigning. $\square$

Claim 2.2. *The matrix M has a signing s such that $M(s)$ is positive semi-definite if and only if there is a ± 1-vector z such that the inner product $\langle b, z \rangle$ is zero.*

Proof. We may assume that any signed matrix $M(s)$ that is positive semi-definite may not have negative entries in the diagonal because a positive semi-definite matrix will not have negative entries on its diagonal. Hence, we will only consider symmetric off-diagonal signing s of the matrix M of the following form:

$$M' := M(s) = \begin{bmatrix} I_n & \hat{b} & z \\ \hat{b}^T & \langle b, b \rangle & 0 \\ z^T & 0 & n \end{bmatrix},$$

where the n-dimensional vector z takes values in $\{\pm 1\}^n$ and $\hat{b} = (\hat{b}_1, \ldots, \hat{b}_n)^T$, where $\hat{b}_i$ takes value in $\{\pm b_i\}$ for every i. Let

$$\begin{aligned} A &:= I_n, \\ B &:= \begin{bmatrix} \hat{b} & z \end{bmatrix}, \text{ and} \\ C &:= \begin{bmatrix} \langle b, b \rangle & 0 \\ 0 & n \end{bmatrix}. \end{aligned}$$

Since $A = I_n$ is invertible, the Schur complement of C in M' is well-defined and is given by

$$\begin{aligned} M'_C &= \begin{bmatrix} \langle b, b \rangle & 0 \\ 0 & n \end{bmatrix} - \begin{bmatrix} \hat{b}^T \\ z^T \end{bmatrix} I_n^{-1} \begin{bmatrix} \hat{b} & z \end{bmatrix} \\ &= \begin{bmatrix} \langle b, b \rangle & 0 \\ 0 & n \end{bmatrix} - \begin{bmatrix} \langle \hat{b}, \hat{b} \rangle & \langle \hat{b}, z \rangle \\ \langle \hat{b}, z \rangle & \langle z, z \rangle \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\langle \hat{b}, z \rangle \\ -\langle \hat{b}, z \rangle & 0 \end{bmatrix}, \end{aligned}$$

where the last equation follows because we have $\langle \hat{b}, \hat{b} \rangle = \langle b, b \rangle$ and $\langle z, z \rangle = n$.

We note that $A = I_n$ is positive definite. Therefore, by property (1) of Lemma 2.1, the matrix M' is positive semi-definite if and only if M'_C is positive semi-definite. Therefore, M' is positive semi-definite if and only if $\langle \hat{b}, z \rangle = 0$. Finally, we note that $\langle \hat{b}, z \rangle = 0$ if and only if there is a ± 1-vector z' such that $\langle b, z' \rangle = 0$. $\square$

Claim 2.3. *The matrix M has a symmetric off-diagonal signing s such that $M(s)$ is singular if and only if there is a vector $z \in \{\pm 1\}^n$ such that the inner product $\langle b, z \rangle$ is zero.*

Proof. Construct the Schur complement M'_C of C in M' as in Claim 2.2. Using property (2) of Lemma 2.1, we have that

$$\det M' = \det(I_n) \cdot \det(M'_C) = \det(I_n) \cdot \det\left(\begin{bmatrix} 0 & -\langle \hat{b}, z\rangle \\ -\langle \hat{b}, z\rangle & 0 \end{bmatrix}\right) = -\langle \hat{b}, z\rangle^2.$$

Therefore, $\det M' = 0$ if and only if $\langle \hat{b}, z\rangle = 0$. Again, we note that $\langle \hat{b}, z\rangle = 0$ if and only if there is a ± 1-vector z' such that $\langle b, z'\rangle = 0$. ☐

To prove Theorem 1.1, we consider the following problem that is closely related to PsdSigning:

NsdSigning: Given a real symmetric matrix M, verify if there exists a signing s such that $M(s)$ is negative semi-definite.

We observe that a real symmetric $n \times n$ matrix is positive semi-definite if and only if $-M$ is negative semi-definite. Theorem 1.2 and this observation lead to the following corollary.

Corollary 2.4. NsdSigning *is NP-complete.*

We next reduce NsdSigning to BoundedEvalueSigning which proves Theorem 1.1.

Proof of Theorem 1.1. BoundedEvalueSigning is in NP since if there is an off-diagonal signing of a given matrix that has all eigenvalues bounded above by a given real number λ, then this signing gives the witness. We can verify if all eigenvalues of a given off-diagonal symmetric signed matrix are at most λ in polynomial time by computing the spectrum of the matrix.

We show NP-hardness of BoundedEvalueSigning by reducing from NsdSigning which is NP-complete by Corollary 2.4. Let the real symmetric $n \times n$ matrix M be the input to the NsdSigning problem. We construct an instance of BoundedEvalueSigning by considering $\lambda = 0$ and the matrix M' obtained from M as follows (where $|a|$ denotes the magnitude of a):

$$M'[i,j] = \begin{cases} M[i,j] & \text{if } i \neq j, \\ -\big|M[i,j]\big| & \text{if } i = j. \end{cases}$$

We observe that every negative semi-definite signing of M has to necessarily have negative values on the diagonal. Hence, there is a signing s such that that $M(s)$ is negative semi-definite if and only if there is an off-diagonal signing t such that $\lambda_{\max}(M'(t)) \leq \lambda = 0$. ☐

3 Matrices with Invertible Signings

In this section, we present a combinatorial characterization of matrices with an invertible signing. We begin with some notation that we use throughout the section.

Let S_n denote the set of permutations of n elements. Then, the *permutation expansion* of the determinant of a signed matrix $M(s)$ is given by

$$\det M(s) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot \prod_{i=1}^{n} M(s)[i, \sigma(i)].$$

For ease of presentation, let us define $M_\sigma(s) \coloneqq \operatorname{sgn}(\sigma) \cdot \prod_i M(s)[i,\sigma(i)]$ and $M_\sigma \coloneqq M_\sigma(J)$, where J is the signing corresponding to all entries being $+1$.

A *matching* is a vertex-disjoint union of edges. A permutation σ in S_n corresponds to a vertex disjoint union of directed cycles and self-loops on n vertices. Removing the orientation gives an undirected graph which is a vertex disjoint union of cycles of length at least three, matching edges, and self-loops. Let the collection of (undirected) edges in the cycle components, matching components, and self-loop components in the resulting undirected graph be denoted by Cycles(σ), Matchings(σ), and Loops(σ) respectively. We observe that sgn(σ) is the *parity* of the sum of the number of matching edges and the number of even length cycles in the undirected subgraph induced by the edges in Cycles(σ) $\cup$ Matchings(σ). We define

$$M_{\text{Cycles}}(\sigma, s) := \left(\prod_{\{u,v\} \in \text{Cycles}(\sigma)} M(s)[u,v] \right),$$

$$M_{\text{Matchings}}(\sigma, s) := \left(\prod_{\{u,v\} \in \text{Matchings}(\sigma)} M(s)[u,v]^2 \right), \text{ and}$$

$$M_{\text{Loops}}(\sigma, s) := \left(\prod_{\{u,u\} \in \text{Loops}(\sigma)} M(s)[u,u] \right).$$

We use the convention that a product over an empty set is equal to 1. With this notation, we have

$$M_\sigma(s) = \text{sgn}(\sigma) \cdot M_{\text{Cycles}}(\sigma, s) \cdot M_{\text{Matchings}}(\sigma, s) \cdot M_{\text{Loops}}(\sigma, s).$$

We now show that every signing of a symmetric matrix is singular if and only if every term in the permutation expansion of the determinant of the matrix is zero.

Lemma 3.1. *Let M be a symmetric matrix. The signed matrix $M(s)$ is singular for every signing s if and only if $M_\sigma = 0$ holds for every permutation σ in S_n.*

Proof. The reverse implication follows immediately: If $M_\sigma = 0$ for every permutation σ in S_n, then for every signing s, $M_\sigma(s) = 0$ for every permutation σ in S_n. Hence, every term in the permutation expansion of the determinant of $M(s)$ is zero for every signing s.

We now show the forward implication. By assumption, the determinant of $M(s)$ is zero for every signing s:

$$\det M(s) = \sum_{\sigma \in S_n} M_\sigma(s) = 0.$$

Assume for the sake of contradiction that there exists a permutation τ with $M_\tau \neq 0$. Let Γ be a *minimum cardinality* subset of permutations in S_n such that

(*i*) $\tau \in \Gamma$ and

(*ii*) $\sum_{\sigma \in \Gamma} M_\sigma(s) = 0$ for every signing s.

We observe that such a set Γ exists since S_n is a valid choice for Γ. The following claim, which we prove later, shows that the cycles and the self-loops of all permutations in Γ coincide with that of τ.

Claim 3.2. *For every permutation σ in Γ, we have Cycles(σ) = Cycles(τ) and Loops(σ) = Loops(τ).*

Now we show that Lemma 3.1 follows from the claim. Let us fix a signing s and a permutation σ in Γ. By Claim 3.2, we have $\text{Cycles}(\sigma) = \text{Cycles}(\tau)$ and $\text{Loops}(\sigma) = \text{Loops}(\tau)$. Furthermore, Claim 3.2 also implies that the number of matching edges in σ and τ is the same and hence, $\text{sgn}(\sigma) = \text{sgn}(\tau)$. So we have

$$\begin{aligned} M_\sigma(s) &= \text{sgn}(\sigma) \cdot M_{\text{Cycles}}(\sigma, s) \cdot M_{\text{Matchings}}(\sigma, s) \cdot M_{\text{Loops}}(\sigma, s) \\ &= \text{sgn}(\tau) \cdot M_{\text{Cycles}}(\tau, s) \cdot M_{\text{Matchings}}(\sigma, s) \cdot M_{\text{Loops}}(\tau, s) \\ &= M_\tau(s) \cdot \left(\frac{M_{\text{Matchings}}(\sigma, s)}{M_{\text{Matchings}}(\tau, s)} \right). \end{aligned}$$

Using the above expression and taking the sum of the terms $M_\sigma(s)$ over all σ in Γ, we have

$$\sum_{\sigma \in \Gamma} M_\sigma(s) = \frac{M_\tau(s)}{M_{\text{Matchings}}(\tau, s)} \cdot \left(\sum_{\sigma \in \Gamma} M_{\text{Matchings}}(\sigma, s) \right).$$

By the choice of τ, we know that $M_\tau(s)/M_{\text{Matchings}}(\tau, s)$ is non-zero. Moreover, $M_{\text{Matchings}}(\sigma, s)$ is a perfect square and therefore positive for all permutations σ in Γ. In particular, since τ is in Γ, we have that Γ is non-empty and hence,

$$\sum_{\sigma \in \Gamma} M_{\text{Matchings}}(\sigma, s) \neq 0.$$

Consequently, the sum $\sum_{\sigma \in \Gamma} M_\sigma(s)$ is non-zero, contradicting condition (ii) in the choice of Γ. □

We now prove Claim 3.2.

Proof of Claim 3.2. Let us consider an arbitrary permutation $\tau' \in \Gamma \backslash \{\tau\}$. Assume for contradiction that there is an edge e in the symmetric difference of $\text{Cycles}(\tau) \cup \text{Loops}(\tau)$ and $\text{Cycles}(\tau') \cup \text{Loops}(\tau')$. Partition Γ into two subsets Γ_e and Γ'_e, where Γ_e contains any permutation σ in Γ that have edge e in the subgraph induced by the edges in $\text{Cycles}(\sigma) \cup \text{Loops}(\sigma)$, and $\Gamma'_e := \Gamma \backslash \Gamma'_e$. By this partitioning, for every permutation $\sigma' \in \Gamma'_e$, either $e \in \text{Matchings}(\sigma')$ or $e \notin \text{Cycles}(\sigma') \cup \text{Matchings}(\sigma') \cup \text{Loops}(\sigma')$ holds. We also observe that exactly one of the permutations τ and τ' is in Γ_e while the other lies in Γ'_e. In particular, both Γ_e and Γ'_e are nonempty. We will show that either Γ_e or Γ'_e contradicts the minimality of Γ.

Let us consider an arbitrary signing s, and let s' be another signing that is obtained from s by flipping the sign only on edge e. Consider the following four sums:

$$\Sigma_{00} := \sum_{\sigma \in \Gamma_e} M_\sigma(s), \quad \Sigma_{01} := \sum_{\sigma \in \Gamma_e} M_\sigma(s'), \quad \Sigma_{10} := \sum_{\sigma \in \Gamma'_e} M_\sigma(s), \quad \Sigma_{11} := \sum_{\sigma \in \Gamma'_e} M_\sigma(s').$$

Now, by condition (ii) in the choice of Γ, we have that

$$\Sigma_{00} + \Sigma_{10} = 0, \tag{1}$$
$$\Sigma_{01} + \Sigma_{11} = 0. \tag{2}$$

For every σ in Γ_e, we have that $M_\sigma(s) = -M_\sigma(s')$ since the edge $e = \{u, v\}$ is in the subgraph induced by the edges in $\text{Cycles}(\sigma) \cup \text{Loops}(\sigma)$ and hence exactly one of the two terms $M(s)[u, v]$ and $M(s)[v, u]$ appears in $M_\sigma(s)$. Therefore,

$$\Sigma_{00} = -\Sigma_{01}. \tag{3}$$

For every σ' in Γ'_e, we have that $M_{\sigma'}(s) = M_{\sigma'}(s')$ since we have either $e \in \text{Matchings}(\sigma')$ or $e \notin \text{Matchings}(\sigma') \cup \text{Cycles}(\sigma') \cup \text{Loops}(\sigma')$ holds and in both cases, an even number of terms among $M(s)[u, v]$ and $M(s)[v, u]$ appear in $M_{\sigma'}(s)$. Therefore,

$$\Sigma_{10} = \Sigma_{11}. \tag{4}$$

By equations (1), (2), (3), and (4), we have that $\Sigma_{00} = \Sigma_{01} = \Sigma_{10} = \Sigma_{11} = 0$ for every signing s.

Now, take T to be the set in $\{\Gamma_e, \Gamma'_e\}$ that contains τ, we obtain that (i) $\tau \in T$ and (ii) $\sum_{\sigma \in T} M_\sigma(s) = 0$ for every signing s. Moreover $|T| < |\Gamma|$, contradicting the minimality of Γ. $\square$

As mentioned in the introduction, we use a classic result by Tutte [47] to prove Theorem 1.5. In fact, we use the following slight generalization of Tutte's result to graphs with self-loops.

Theorem 3.3 (Tutte [47], also see Lovász and Plummer [32])**.** *A graph G (possibly with self-loops) has no perfect 2-matching if and only if G has a non-empty independent set Q with $|N(Q)| < |Q|$ where none of the vertices in Q have self-loops. Moreover, there is a polynomial-time algorithm to verify if a given graph has a perfect 2-matching.*

For the sake of completeness, we present a proof of Theorem 3.3 in the appendix (see §A). We now prove Theorem 1.5.

Proof of Theorem 1.5. By Lemma 3.1, the signed matrix $M(s)$ is singular for every signing s if and only if $M_\sigma = 0$ holds for every permutation σ in S_n. The existence of a perfect 2-matching in the support graph of M is equivalent to the fact that $M_\sigma \neq 0$ for some σ in S_n, and therefore we have that $M_\sigma = 0$ for every σ in S_n if and only if the support graph of M has no perfect 2-matchings. By Theorem 3.3, a graph G has no perfect 2-matching if and only if G has a non-empty independent set Q with $|N(Q)| < |Q|$ where none of the vertices in Q have self-loops.

The polynomial-time algorithm given in Theorem 3.3 immediately gives us a polynomial-time algorithm to verify whether the signed matrix $M(s)$ is singular for every signing s. $\square$

4 Minimum Support Increase to Obtain an Invertible Signing

In this section, we study the problem of computing the solvability index of real symmetric matrices, thus proving Theorem 1.6. We recall the following definition: For a real symmetric matrix M, the *solvability index* of M is the smallest number of non-diagonal zero entries that need to be converted to non-zeroes so that the resulting *symmetric* matrix has an invertible signing. (Be reminded that the support-increase operation preserves symmetry.)

By our characterization in Theorem 1.5 and the results of Tutte summarized in Theorem 3.3, computing the solvability index of a matrix reduces to the following edge addition problem:.

EdgeAdd: Given a graph G (possibly with self-loops) with vertex set V and edge set E, find

$$\min \left\{ |F| \;\middle|\; F \text{ is a set of non-edges of } G \text{ such that } G + F \text{ has a perfect 2-matching} \right\}.$$

In the above, $G + F$ denotes the graph obtained by adding the edges in F to G. In the rest of the section, we will show that EdgeAdd can be solved efficiently, which will imply Theorem 1.6.

Theorem 4.1. *There is a polynomial time algorithm to solve* EdgeAdd.

Preliminaries. We need some terminology from matching theory. Let G be a graph on vertex set V and edge set E. For a subset S of vertices, denote the *induced subgraph* of G on S as $G[S]$ and the *non-inclusive neighborhood* of S in G by $N_G(S)$. We recall that a *matching* M in G is a subset of edges where each vertex is incident to at most one edge in M. Let $\nu(G)$ denote the cardinality of a *maximum matching* in G and let

$$\nu_f(G) := \max\left\{\sum_{e\in E} x_e \,\middle|\, \sum_{e\in\delta(v)} x_e \le 1, \text{ and } x_e \ge 0 \text{ for all } e \in E\right\}$$

denote the value of a *maximum fractional matching* in G. For a matching M, we define a vertex u to be M*-exposed* if none of the edges of M are incident to u, and a vertex v to be an M*-neighbor* of u if edge $\{u, v\}$ is in M. A vertex u in V is said to be *inessential* if there exists a maximum cardinality matching M in G such that u is M-exposed, and is said to be *essential* otherwise. A graph H is *factor-critical* if there exists a perfect matching in $H - v$ for every vertex v in H. The following result is an immediate consequence of the odd-ear decomposition characterization of Lovász [31].

Lemma 4.2 (Lovász [31])**.** *If G is a factor-critical graph, then G has a perfect 2-matching.*

The *Gallai-Edmonds decomposition* [14, 17, 18] of a graph G is a partition of the vertex set of G into three sets (B, C, D), where B is the set of inessential vertices, $C := N_G(B)$, and $D := V \setminus (B \cup C)$. Let B_1 denote the set of isolated vertices in $G[B]$ and $B_{\geq 3} := B \setminus B_1$. For notational convenience, we will denote the Gallai-Edmonds decomposition as $(B = (B_1, B_{\geq 3}), C, D)$. The Gallai-Edmonds decomposition of a graph is unique and can be found efficiently [14]. The following theorem summarizes the properties of the Gallai-Edmonds decomposition that we will be using (properties (i) and (ii) are well-known and can be found in Schrijver [43] while property (iii) follows from results due to Balas [7] and Pulleyblank [40]—see Bock *et al.* [9] for a proof of property (iii)):

Theorem 4.3. *Let $(B = (B_1, B_{\geq 3}), C, D)$ be the Gallai-Edmonds decomposition of a graph G. We have the following properties:*

(i) Each connected component in $G[B]$ is factor-critical.

(ii) Every maximum matching M in G contains a perfect matching in $G[D]$ and matches each vertex in C to distinct components in $G[B]$.

(iii) Let M be a maximum matching that matches the largest number of B_1 vertices. Then there are $2(\nu_f(G) - \nu(G))$ M-exposed vertices in $B_{\geq 3}$.

Observe that G contains a perfect 2-matching if and only if $\nu_f(G) = |V|/2$. Therefore, adding edges to get a perfect 2-matching in G is equivalent to adding edges to increase the maximum fractional matching value to $|V|/2$.

Proof of Theorem 4.1. We will assume that G has no isolated vertices and no self-loops in the rest of the proof. We make this assumption here in order to illustrate the main idea underlying the algorithm. This assumption can be relaxed by a case analysis in the algorithm as well as the proof of correctness. We defer the details of the case analysis to the full-version of the paper.

We use algorithm EdgeAdd(G) given in Figure 4.1. We briefly describe an efficient implementation for Step 2, since it is easy to see that other steps can be implemented efficiently. In order to find a maximum matching that matches the largest number of B_1 vertices (as mentioned in property (iii) of Theorem 4.3), we first find the Gallai-Edmonds decomposition and a maximum matching

<u>EDGEADD(G):</u> *Input*: A graph G with no isolated vertices and no self-loops.
1. Find the Gallai-Edmonds decomposition $(B = (B_1, B_{\geq 3}), C, D)$ of G.
2. Find a maximum matching M that matches the largest number of B_1 vertices.
3. Let $S := \{u \in B_1 \mid u \text{ is } M\text{-exposed}\}$.
4. If $\lvert S\rvert$ is even:
    Pick an arbitrary pairing of the vertices in S.
5. If $\lvert S\rvert$ is odd:
    Consider a vertex s in S, pick a vertex t in $N_G(s)$ and let u be the M-neighbor of t.
    Pair up u with s and pick an arbitrary pairing of the vertices in $S \setminus \{s\}$.
6. Return the set of pairs F.

Figure 4.1: The algorithm EDGEADD(G).

M. Then, we repeatedly augment M by searching for M-alternating paths (of even-length) from M-exposed B_1 vertices. This approach can be implemented to run in polynomial time. Alternatively, Step 2 can also be implemented by solving a maximum weight matching with suitably chosen weights.

We now argue the correctness of the algorithm. We first show that if $|S|$ is odd, then there is a choice of vertices t and u as described in the algorithm EDGEADD(G): this is because, G has no isolated vertices and hence there exists a vertex t in $N_G(s)$. Moreover, by Theorem 4.3, since s is in B_1, it follows that t is in C and thus t is matched by M to a node u in B. Now, Claim 4.4 proves feasibility and bounds the size of the returned solution F while Claim 4.5 proves the optimality. □

Claim 4.4. *The algorithm* EDGEADD(G) *returns a set F of non-edges of G such that (1) $G + F$ contains a perfect 2-matching, and (2) $|F| = \lceil |V|/2 - \nu_f(G) \rceil$.*

Proof. By property (ii) of Theorem 4.3, the set F is a set of non-edges of G. We will construct a perfect 2-matching in $G+F$. By property (i) of Theorem 4.3, every component in $G[B_{\geq 3}]$ is factor-critical. By Lemma 4.2, every component K in $G[B_{\geq 3}]$ contains a perfect 2-matching x^K. Let N_K denote the support of x^K. Let $\mathcal{K}$ denote the components in $G[B_{\geq 3}]$ that contain an M-exposed vertex. We have two cases:

Case 1: Suppose $|S|$ is even. Let N denote the set of edges of M that do not match any vertices in $\bigcup_{K \in \mathcal{K}} V(K)$. Now, the set of edges induced by $\left(\bigcup_{K\in\mathcal{K}} N_K\right) \cup N \cup F$ has a perfect 2-matching. A perfect 2-matching x in $G+F$ can be obtained by assigning $x(e) := x^K(e)$ for edges e in $\bigcup_{K\in\mathcal{K}} N_K$, $x(e) := 2$ for edges e in $N \cup F$, and $x(e) := 0$ for the remaining edges in $G + F$.

Case 2: Suppose $|S|$ is odd. Let N denote the set of edges of $M \setminus \{\{t, u\}\}$ that do not match any vertices in $\bigcup_{K\in\mathcal{K}} V(K)$. Now, $\left(\bigcup_{K\in\mathcal{K}} N_K\right) \cup N \cup (F \setminus \{s, u\}) \cup \{\{t, u\}, \{s, t\}, \{s, u\}\}$ has a perfect 2-matching. We note that the edges $\{t, u\}, \{s, t\}$ were already present in the graph owing to the choice of c and u while the edge $\{s, u\}$ was added as an edge from F. A perfect 2-matching x in $G + F$ can be obtained by assigning $x(e) := x^K(e)$ for edges $e \in \bigcup_{K\in\mathcal{K}} N_K$, $x(e) := 1$ for edges e in $\{\{t, u\}, \{s, t\}, \{s, u\}\}$, $x(e) := 2$ for edges e in $N \cup (F \setminus \{s, u\})$, and $x(e) := 0$ for the remaining edges in $G + F$.

Next we find the size of the set F returned by the algorithm. We observe that $|F| = \lceil |S|/2 \rceil$. It remains to bound $|S|$. For this, we count the number of vertices in the graph using the matched and exposed vertices. We have that $|V| = 2|M| + |S| + \{\text{number of } M\text{-exposed vertices in } B_{\geq 3}\}$.

By property (iii) of Theorem 4.3 and the choice of the matching M, we have $|V| = 2|M| + |S| + 2(\nu_f(G) - \nu(G))$. Since M is a maximum cardinality matching, we know that $|M| = \nu(G)$ and hence, $|S| = |V| - 2\nu_f(G)$. $\square$

Our next claim shows a lower bound on the optimal solution that matches the upper bound and hence proves the optimality of the returned solution.

Claim 4.5. *Let F' be a set of non-edges of G. Suppose $G + F'$ has a perfect 2-matching. Then $|F'| \geq \lceil |V|/2 - \nu_f(G) \rceil$.*

Proof. We first note that the addition of a non-edge can increase the value of the maximum fractional matching by at most one. That is, for every graph H and every non-edge e of H, we have $\nu_f(H + e) - \nu_f(H) \leq 1$ (this can be shown by considering the dual problem, namely the minimum fractional vertex cover). Now, consider an arbitrary ordering of the edges in the solution F' and let F'_i denote the set of first i edges according to this order and let $F'_0 = \emptyset$. Then,

$$\nu_f(G + F') - \nu_f(G) = \sum_{i=1}^{|F'|} (\nu_f(G + F_i) - \nu_f(G + F_{i-1})) \leq |F'|.$$

Thus, we have $|F'| \geq \nu_f(G + F') - \nu_f(G)$. We observe that if $G + F'$ has a perfect 2-matching, then $\nu_f(G + F') = |V|/2$. Hence, $|F'| \geq |V|/2 - \nu_f(G)$. Finally, we observe that $|F'|$ has to be an integer and hence, $|F'| \geq \lceil |V|/2 - \nu_f(G) \rceil$. $\square$

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A Appendix

Proof of Theorem 3.3. Let G be a graph (possibly with self-loops) on n vertices. Consider a bipartite graph H with two parts L and R constructed as follows. For every vertex v of G we create vertices $v_L \in L$ and $v_R \in R$ in H. Moreover, for every edge $\{v, u\}$ of G we create edges $\{v_L, u_R\}$ and $\{u_L, v_R\}$ in H and for every self-loop edge $\{v, v\}$ of G we create edge $\{v_L, v_R\}$.

Claim A.1. *G has a perfect 2-matching if and only if H has a perfect matching.*

Proof. Let E be the set of edges in G and $x : E \to \{0, 1, 2\}$ be a perfect 2-matching in G. Then consider the support M of x. The graph $G[M]$ is a spanning subgraph of G where every connected component is either a cycle, an edge, or a self-loop. We fix an orientation D of the edges of $G[M]$ such that each cycle component is strongly connected. Consider the set

$$M' \coloneqq \{\{u_L, v_R\} \mid (u, v) \in D\}.$$

Then M' is a subset of edges in H. Since D is a vertex-disjoint union of directed cycles, directed edges, and self-loops, M' is a perfect matching in H.

Likewise, if the edges M' in H is a perfect matching in H, then we consider the function

$$x'(\{u, v\}) \coloneqq \begin{cases} 2 & \text{if both } \{u_L, v_R\} \text{ and } \{u_L, v_R\} \text{ are in } M, \\ 1 & \text{if exactly one of } \{u_L, v_R\} \text{ and } \{u_L, v_R\} \text{ is in } M, \\ 0 & \text{otherwise} \end{cases}$$

which is a perfect 2-matching in G. □

By the equivalence in Claim A.1 we can now determine if a graph has a perfect 2-matching by constructing the bipartite graph H as above and searching for a perfect matching in H can be done in polynomial time.

Suppose G has a perfect 2-matching. It follows from Claim A.1 that H must have a perfect matching. Hence, by Hall's theorem, every subset of vertices of H must be expanding in H. This implies that any subset Q of vertices in G must be expanding in G, including independent sets without self-loops.

Suppose that G has no perfect 2-matching. Then H has no perfect matching. By Hall's theorem there must be a non-expanding subset $S' \subseteq L$. Let $S \coloneqq \{u \mid u_L \in S'\}$. Since S' is not expanding in H, we have that S is not expanding in G. We will show that S must be an independent set in G without self-loops. Suppose to the contrary that S is either not an independent set or contains self-loops. Let Q be the largest independent set in S without self-loops. Let $Q' \coloneqq \{u_R \mid u \in Q\}$ and $\overline{Q'} \coloneqq \{u_R \mid u \in S \setminus Q\}$. It follows that $Q' \cap \overline{Q'} = \emptyset$ by construction. We observe every vertex in S that is not in Q must have a neighbor in Q or must have a self-loop. Hence, $Q' \cup \overline{Q'} \subseteq N_H(S')$ holds. However, $\left|Q' \cup \overline{Q'}\right| = |S| = |S'|$. Thus, $|S'| \leq |N_H(S')|$ and S' must be expanding in H, a contradiction to the choice of S'. □