REGULARITY OF POWERS OF UNICYCLIC GRAPHS

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Abstract. Let G be a finite simple graph and $I(G)$ denote the corresponding edge ideal. In this paper we prove that if G is a unicyclic graph then for all $s \geq 1$ the regularity of $I(G)^s$ is exactly $2s + \text{reg}(I(G)) - 2$. We also characterize the unicyclic graphs with regularity $\nu(G)+1$ and $\nu(G)+2$, where $\nu(G)$ denotes the induced matching number of G.

1. Introduction

Let $G = (V(G), E(G))$ denote a finite simple (no loops, no multiple edges) undirected graph with vertices $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$. By identifying the vertices with the variables in the polynomial ring $k[x_1, \ldots, x_n]$, where k is a field, we can associate each graph G to a monomial ideal $I(G)$ generated by the set $\{x_i x_j \mid \{x_i, x_j\} \in E(G)\}$. The ideal $I(G)$ is called the *edge ideal* of G. Recently, building a dictionary between combinatorial data of graphs and the algebraic properties of the corresponding edge ideals has been studied by various authors, see [BHT15], [BC13], [Frö90], [Hà14], [HVT08], [HHKT16], [Jac04], [JNS16], [Kat06], [MFY17], [Mor10], [Woo14], [Zhe04]. In particular, establishing a relationship between Castelnuovo-Mumford regularity of the edge ideals and combinatorial invariants associated with graphs such as induced matching number, matching number and co-chordal cover number is an active research topic, see [HVT08], [HHKT16], [Kat06], [Woo14].

Our motivation to study regularity of powers of edge ideals springs from a famous result: for a homogeneous ideal I in a polynomial ring, $\text{reg}(I^s)$ is asymptotically a linear function for $s \gg 0$, (cf. [CHT99], [Kod00], [TW05], [Cha07]), i.e., there exist integers a, b, s_0 such that

$$\text{reg}(I^s) = as + b \text{ for all } s \geq s_0.$$

In this regard, there has been an interest in finding the exact form of the linear function and determining the stabilization index s_0 where $\text{reg}(I^s)$ becomes linear (cf. [Ber12], [Cha13], [EH10], [EU12], [Hà11]). It turns out that even in the case of monomial ideals it is challenging to find the linear function and s_0 (cf. [Con06], [Hà14]). In this paper, we consider $I = I(G)$, the edge ideal of G. In this case, there exist integers b and s_0 such that $\text{reg}(I^s) = 2s + b$ for all $s \geq s_0$. There are very few classes of graphs for which b and s_0 are explicitly computed. We refer the reader to [BHT15] and [JNS16] for a review of results in the literature which identify classes of edge ideals for which b and s_0 are known. Our aim in this paper is to find b and s_0 in terms of combinatorial invariants of the graph G when G is a unicyclic graph.

A *unicyclic graph* G is a connected graph containing exactly one cycle. If G is a cycle, then its regularity is known due to Jacques [Jac04, Theorem 7.6.28] and the linear function

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to describe regularity of powers of cycles is given in [BHT15, Theorem 5.2]. Throughout the paper, we shall restrict our attention to unicyclic graphs which are not cycles.

The first step of computing regularity of powers of unicyclic graphs, $\operatorname{reg}(I(G)^s)$ for all $s \geq 1$, requires us to compute $\operatorname{reg}(I(G))$. It is known that for any unicyclic graph G,

$$\nu(G) + 1 \leq \operatorname{reg}(I(G)) \leq \nu(G) + 2,$$

where $\nu(G)$ denote the induced matching number of G. The lower bound was proved by Katzman, [Kat06] and the upper bound was proved by Bıyıkoğlu and Civan, [BC13]. Section 3 is where to find a characterization of unicyclic graphs G with $\operatorname{reg}(I(G)) = \nu(G) + 1$ and $\operatorname{reg}(I(G)) = \nu(G) + 2$. We provide the characterization of unicyclic graphs with regularity $\nu(G) + 2$. Our result is stated as follows.

Theorem 1.1. (Corollary 3.13.) *Let G be a unicyclic graph with cycle C_n. Then we have $\operatorname{reg}(I(G)) = \nu(G) + 2$, if and only if the following conditions are satisfied:*

(1) $n \equiv 2 (mod\ 3)$.
(2) *C_n is a connected component of $G \setminus S$, where S is the 2-special vertex set of G.*
(3) $\nu(G \setminus S) = \nu(G)$.

In order to prove Theorem 1.1, we provide necessary conditions for a unicyclic graph to have regularity $\nu(G) + 1$, (Lemmas 3.8, 3.9, 3.11, 3.12) and $\nu(G) + 2$ (Theorem 3.6). The characterization of unicyclic graphs with regularity $\nu(G) + 1$ (Corollary 3.15) follows from Theorem 1.1.

We then move on to compute precise expressions for the regularity of powers of edge ideals of unicyclic graphs. In this paper, we prove that asymptotic regularity of a unicyclic graph G can be given in terms of regularity of G. More interestingly, we add a new linear function to describe asymptotic regularity of powers of edge ideals and it is equal to $2s + \nu(G)$ when $\operatorname{reg}(I(G)) = \nu(G) + 2$. The main result of the paper is the following:

Theorem 1.2. (Theorem 5.3.) *If G is a unicyclic graph, then for all $s \geq 1$,*

$$\operatorname{reg}(I(G)^s) = 2s + \operatorname{reg}(I(G)) - 2.$$

Note that for this class of graphs, we have $b = \operatorname{reg}(I(G)) - 2$ and $s_0 = 1$. As an immediate consequence, we derive one of the main results of [MFY17], that the above equality holds for whiskered cycle graphs.

To prove Theorem 1.2, we establish the upper bound $\operatorname{reg}(I(G)^s) \leq 2s + \operatorname{reg}(G) - 2$ for all $s \geq 1$ when G is a unicyclic graph (Lemma 5.2). This upper bound coupled with the lower bound given in [BHT15, Theorem 4.5] leads us to the following:

$$2s + \nu(G) - 1 \leq \operatorname{reg}(I(G)^s) \leq 2s + \operatorname{reg}(G) - 2.$$

It follows from the above inequalities that $\operatorname{reg}(I(G)^s) = 2s + \nu(G) - 1$ for all $s \geq 1$ when $\operatorname{reg}(I(G)) = \nu(G) + 1$. In the case where $\operatorname{reg}(I(G)) = \nu(G) + 2$, we present an induced subgraph of G, say H, such that $\operatorname{reg}(I(H)^s) = 2s + \nu(G)$. Thus by making use of [BHT15, Corollary 4.3] and the upper bound, we prove that $\operatorname{reg}(I(G)^s) = 2s + \nu(G)$ for all $s \geq 1$.

Our paper is organized as follows. In section 2, we collect the necessary notation and terminology that will be used in the paper. In Section 3, we prove Theorem 1.1. Section 4 is devoted for finding bounds for regularity of special colon ideals to build the proof of our main result in Section 5. Finally, we prove Theorem 1.2 in Section 5.

2. Preliminaries

In this section, we set up the basic definitions and terminology needed for the main results. Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex x in a graph G, let $N_G(x) = \{y \in V(G) \mid \{x, y\} \in E(G)\}$ be the set of neighbors of x and set $N_G[x] = N_G(x) \cup \{x\}$. An edge e is *incident* to a vertex x if $x \in e$. The *degree* of a vertex $x \in V(G)$, denoted by $\deg_G(x)$, is the number of edges incident to x. If $\deg_G(x) = 1$, then x is called a *leaf*, a *free vertex*, or a *pendant vertex* of G. If x is a leaf and $N_G(x) = \{y\}$, let $e = \{x, y\}$, then we also call the edge e a *leaf* of G. Let C_n denote the cycle on n vertices and P_n denote the path on n vertices. The length of a path, or a cycle is its number of edges.

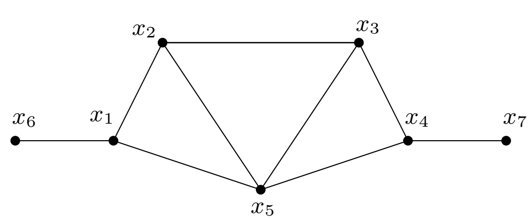
Letting $e \in E(G)$, then define $G \setminus e$ to be the subgraph of G obtained from G by deleting the edge e but keeping its vertices. If $W \subseteq V(G)$ in G, then $G \setminus W$ denotes the subgraph of G with the vertices in W and all incident edges deleted. When $W = \{x\}$ consists of a single vertex, we shall write $G \setminus x$ instead of $G \setminus \{x\}$. If $e = \{x, y\}$, then set $N_G[e] = N_G[x] \cup N_G[y]$ and define G_e to be the subgraph $G \setminus N_G[e]$ of G.

A graph H is called an *induced subgraph* of G if the vertices of H are the vertices of G, and for the vertices x and y in H, $\{x, y\}$ is an edge in H if and only if $\{x, y\}$ is an edge in G. The induced subgraph of G over a subset $W \subseteq V(G)$ is obtained by deleting all the vertices that are not in W from G.

Let G and H be graphs. Then we denote the union of two graphs with $G \cup H$ and $V(G \cup H) := V(G) \cup V(H)$ and $E(G \cup H) := E(G) \cup E(H)$. If G and H disjoint graphs(i.e., $V(G) \cap V(H) = \emptyset$), we denote the disjoint union of G and H by $G \coprod H$.

A *matching* in a graph G is a collection of pairwise disjoint edges $\{e_1, \ldots, e_s\}$. We call a collection of edges $\{e_1, \ldots, e_s\}$ an *induced matching* if they form a matching in G, and they are exactly the edges of the induced subgraph of G over the vertices $\bigcup_{i=1}^n e_i$. The largest size of an induced matching in G is called its *induced matching number* and denoted by $\nu(G)$. Note that if H is an induced subgraph of G, then $\nu(H) \leq \nu(G)$. Furthermore, if G and H are disjoint graphs, then $\nu(G \coprod H) = \nu(G) + \nu(H)$.

Example 2.1. Let G be a graph with $V(G) = \{x_1, \ldots, x_7\}$.



Then $\{x_1x_6, x_2x_3, x_4x_7\}$ forms a matching, but not an induced matching (the induced subgraph on $\{x_1, x_2, x_3, x_4, x_6, x_7\}$ also contains edges $\{x_1x_2, x_3x_4\}$). The induced matching number $\nu(G)$ is 2.

Definition 2.2. Let R be a standard graded polynomial ring over a field k. The *Castelnuovo-Mumford regularity* (or regularity) of a finitely generated graded R module M, written $\operatorname{reg}(M)$ is given by

$$\operatorname{reg}(M) := \max\{j - i \mid \operatorname{Tor}_i(M, k)_j \neq 0\}.$$

Let I be a non-zero proper homogeneous ideal of R. Then it is straight from the definition that $\operatorname{reg}(R/I) = \operatorname{reg}(I) - 1$.

We use the following well-known theorem to prove an upper bound for the regularity of edge ideals inductively:

Theorem 2.3. [Hà14, Lemma 3.1, Theorem 3.4 and 3.5] *Let $G = (V(G), E(G))$ be a graph.*

(1) *If H is an induced subgraph of G, then* $\operatorname{reg}(I(H)) \leq \operatorname{reg}(I(G))$.
(2) *Let $x \in V(G)$. Then*

$$\operatorname{reg}(I(G)) \leq \max\{\operatorname{reg}(I(G \setminus x)), \operatorname{reg}(I(G \setminus N[x])) + 1\}.$$

(3) *Let $e \in E(G)$. Then*

$$\operatorname{reg}(I(G)) \leq \max\{\operatorname{reg}(I(G \setminus e)), \operatorname{reg}(I(G_e)) + 1\}.$$

For a monomial $M \in R = k[x_1, \dots, x_n]$, *support of M* is the set of variables appearing in M and is denoted by $\operatorname{supp}(M)$, i.e., $\operatorname{supp}(M) = \{x_i \mid x_i \text{ divides } M\}$.

We recall the definition of even-connectedness and its important properties from [Ban15].

Definition 2.4. Let $G = (V(G), E(G))$ be a graph. Two vertices u and v (u may be the same as v) are said to be even-connected with respect to an s-fold product $e_1 \cdots e_s$, where e_i's are edges of G, not necessarily distinct, if there is a path $p_0 p_1 \cdots p_{2k+1}$, $k \geq 1$ in G such that:

(1) $p_0 = u, p_{2k+1} = v$.
(2) For all $0 \leq l \leq k-1$, $p_{2l+1}p_{2l+2} = e_i$ for some i.
(3) For all i, $\mid \{l \geq 0 \mid p_{2l+1}p_{2l+2} = e_i\} \mid \leq \mid \{j \mid e_j = e_i\} \mid$.
(4) For all $0 \leq r \leq 2k$, $p_r p_{r+1}$ is an edge in G.

Example 2.5. In Example 2.1 if we set $e_1 = x_1x_5$ and $e_2 = x_3x_4$, then we have x_6 and x_7 are even-connected in G with respect to e_1e_2 since we have the path $(p_0 = x_6)x_1x_5x_3x_4(x_7 = p_5)$. Also note that x_2 is even-connected to itself with respect to e_1e_2 since we have the path $x_2x_1x_5x_4x_3x_2$.

The following theorem is used repeatedly throughout this paper.

Theorem 2.6. [Ban15, Theorem 6.1 and Theorem 6.7] *Let G be a graph with edge ideal $I = I(G)$, and let $s \geq 1$ be an integer. Let M be a minimal generator of I^s. Then $(I^{s+1} : M)$ is minimally generated by monomials of degree 2, and uv (u and v may be the same) is a minimal generator of $(I^{s+1} : M)$ if and only if either $\{u, v\} \in E(G)$ or u and v are even-connected with respect to M.*

Polarization is a process to obtain a squarefree monomial ideal from a given monomial ideal and it behaves well under regularity. For details of polarization we refer to [Far06].

Definition 2.7. Let $M = x_1^{a_1} \dots x_n^{a_n}$ be a monomial in $R = k[x_1, \dots, x_n]$. Then we define the squarefree monomial $P(M)$ (*polarization* of M) as

$$P(M) = x_{11} \dots x_{1a_1} x_{21} \dots x_{2a_2} \dots x_{n1} \dots x_{na_n}$$

in the polynomial ring $S = k[x_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq a_i]$. If $I = (M_1, \dots, M_q)$ is an ideal in R, then the polarization of I, denoted by $\widetilde{I}$, is define as $\widetilde{I} = (P(M_1), \dots, P(M_q))$.

Corollary 2.8. [HH11, Corollary 1.4.9] *Let I be a monomial ideal in $k[x_1, \dots, x_n]$. Then*

$$\operatorname{reg}(I) = \operatorname{reg}(\widetilde{I}).$$

Let G be a graph and let $I(G)$ denote the edge ideal of G. Then for any $s \geq 1$ and edges $e_1, \dots, e_s$ of G, $\widetilde{I} = \widetilde{(I(G)^{s+1} : e_1 \cdots e_s)}$ is a squarefree quadratic monomial ideal, by Theorem 2.6. Hence there exists a graph G' associated to $\widetilde{I}$. Note also that G is a subgraph of G'.

Example 2.9. Let G be the graph of the Example 2.1 and

$$I(G) = (x_1x_6, x_1x_2, x_1x_5, x_2x_5, x_2x_3, x_3x_5, x_5x_4, x_4x_3, x_4x_7) \subset K[x_1, \ldots, x_7].$$

Then $I = (I(G)^2 : x_2x_3) = I(G) + (x_3^2, x_3x_4)$. Therefore, $\widetilde{I} \subset k[x_1, \ldots, x_7, y_1]$ is given by $\widetilde{I} = I(G) + (x_5y_1, x_1x_4)$.

3. Regularity of unicyclic graphs

In [BC13], the authors studied the regularity of unicyclic graphs. In this section, we provide the characterization of unicyclic graphs with regularity $\nu(G) + 1$ and regularity $\nu(G) + 2$.

3.1. **Notation.** We use the following notation for the rest of the paper:

Let G be a unicyclic graph with cycle C_n. Let F be the graph with $E(F) = E(G) \setminus E(C_n)$ and $\Gamma_3(G)$ denote the set of all paths Q in G such that

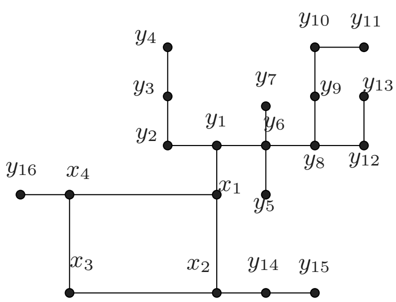
(1) $Q : z_1z_2 \cdots z_{3m+1}$ is a path of length $3m$ for some $m \geq 1$,
(2) $z_1 \in V(C_n) \cap V(F)$,
(3) z_{3m+1} is a leaf in F,
(4) $z_i \in V(F) \setminus V(C_n)$ for all $2 \leq i \leq 3m + 1$.

and let S be defined by

$$S = \{z_{3r-1} \in V(Q) \mid 1 \leq r \leq m,\ Q \in \Gamma_3(G)\}.$$

We shall call $\Gamma_3(G)$ the *3-special path set* of G and S the *2-special vertex set* of G.

Example 3.1. Let G denote the given graph on vertices $\{x_1, \ldots, x_4, y_1, \ldots, y_{16}\}$:



Let $P : x_1y_1y_6y_7$, $P' : x_1y_1y_6y_5$ and $P'' : x_1y_1y_6y_8y_9y_{18}y_{11}$ be induced paths in G. Then $\Gamma_3(G) = \{P, P', P''\}$ and $S = \{y_1, y_9\}$.

The following theorem by Bıyıkoğlu and Civan turns out to be crucial in proving our main results.

Theorem 3.2. [BC13, Corollary 4.12] *If G is a unicyclic graph, then*

$$\nu(G) + 1 \leq \operatorname{reg}(I(G)) \leq \nu(G) + 2.$$

The following observation will be used repeatedly in our proofs.

Observation 3.3. Let G be a graph with a leaf u and its unique neighbor v, say $e = \{u, v\}$. If $\{e_1, \ldots, e_s\}$ is an induced matching in $G \setminus N_G[v]$, then $\{e_1, \ldots, e_s, e\}$ is an induced matching in G. Therefore, $\nu(G \setminus N_G[v]) + 1 \leq \nu(G)$.

The following lemma is frequently needed in the sequel.

Lemma 3.4. *Let G be obtained from $C_n : x_1x_2 \cdots x_nx_1$ by attaching a pendant, say y, to a vertex x_i. Then*

$$\operatorname{reg}(I(G)) = \nu(G) + 1.$$

Proof. By [BC15, Lemma 3.25], we have

$$\operatorname{reg}(I(G)) = \operatorname{reg}(I(G \setminus y)) \quad \text{or} \quad \operatorname{reg}(I(G)) = \operatorname{reg}(I(G \setminus N_G[x_i])) + 1.$$

If $\operatorname{reg}(I(G)) = \operatorname{reg}(I(G \setminus y))$, then by [Jac04, Theorem 7.6.28]

$$\operatorname{reg}(I(G)) = \operatorname{reg}(I(G \setminus y)) = \operatorname{reg}(I(C_n)).$$

If $n \equiv \{0,1\}(mod\ 3)$, then $\operatorname{reg}(I(G)) = \operatorname{reg}(I(C_n)) = \nu(G) + 1$. If $n \equiv 2(mod\ 3)$, then $\operatorname{reg}(I(G)) = \operatorname{reg}(I(C_n)) = \nu(C_n) + 2 = \nu(G) + 1$. If $\operatorname{reg}(I(G)) = \operatorname{reg}(I(G \setminus N_G[x_i])) + 1$, then by [Zhe04, Theorem 2.18] and Observation 3.3,

$$\operatorname{reg}(I(G)) = \operatorname{reg}(I(G \setminus N_G[x_i])) + 1 = \nu(G \setminus N_G[x_i]) + 2 \leq \nu(G) + 1.$$

By [Kat06, Lemma 2.2], we have $\operatorname{reg}(I(G)) = \operatorname{reg}(I(G \setminus N_G[x_i])) + 1 = \nu(G) + 1$. Therefore $\operatorname{reg}(I(G)) = \nu(G) + 1$. □

Definition 3.5. Let $G = (V(G), E(G))$ be a graph and $H = (V(H), E(H))$ be a subgraph of G. We shall call H is a *connected component* of G if H is connected and for all vertices x such that $x \in V(G)$ and $x \notin V(H)$ there is no vertex $y \in V(H)$ such that $\{x, y\} \in E(G)$.

We now obtain some sufficient conditions for $\operatorname{reg}(I(G)) = \nu(G) + 2$ when G is a unicyclic graph.

Theorem 3.6. *Let G be a unicyclic graph with cycle C_n. If all the following conditions hold*

(1) $n \equiv 2(mod\ 3)$,
(2) *C_n is a connected component of $G \setminus S$, where S is the 2-special vertex set of G,*
(3) $\nu(G \setminus S) = \nu(G)$,

then $\operatorname{reg}(I(G)) = \nu(G) + 2$.

Proof. Since C_n is a connected component of $G \setminus S$, $G \setminus S = C_n \coprod (\coprod_{i=1}^{t} H_i)$, where H_i's are trees. Note that $\nu(G \setminus S) = \nu(C_n) + \nu(\coprod_{i=1}^{t} H_i)$. Then

$$\begin{aligned}
\operatorname{reg}(I(G \setminus S) &= \operatorname{reg}(I(C_n)) + \operatorname{reg}(I(\textstyle\coprod_{i=1}^{t} H_i)) - 1 && \text{(by [Woo14, Lemma 8])} \\
&= \nu(C_n) + 2 + \nu(\textstyle\coprod_{i=1}^{t} H_i)) && \text{(by [Jac04, Theorem 7.6.28])} \\
&= \nu(G \setminus S) + 2 \\
&= \nu(G) + 2.
\end{aligned}$$

Since $G \setminus S$ is an induced subgraph of G, then by Theorem 2.3 and Theorem 3.2, we get the desired equality $\operatorname{reg}(I(G)) = \nu(G) + 2$. □

Set-up 3.7. Let G be a unicyclic graph with cycle C_n and F be the graph with the edge set $E(F) = E(G) \setminus E(C_n) = \{f_1, \ldots, f_k\}$.

In the following lemmas 3.8, 3.9, 3.11 and 3.12, we give some sufficient conditions for $\operatorname{reg}(I(G)) = \nu(G) + 1$ when G is a unicycle graph.

Lemma 3.8. *(with notation as in Set-up 3.7). If we assume $n \equiv \{0, 1\}(mod\ 3)$, then we have* $\operatorname{reg}(I(G)) = \nu(G) + 1$.

Proof. By [Kat06, Lemma 2.2], we have $\operatorname{reg}(I(G)) \geq \nu(G) + 1$. It suffices to show that $\operatorname{reg}(I(G)) \leq \nu(G) + 1$. We use induction on k. If $k = 1$, then by Lemma 3.4, we have

$\operatorname{reg}(I(G)) = \nu(G)+1$. Assume that $k \geq 2$. There is a leaf y in G such that $\{x\} = N_G(y)$. Set $G' = G \setminus x$ and $G'' = G \setminus N_G[x]$. By Theorem 2.3, we have

$$\operatorname{reg}(I(G)) \leq \max\{\operatorname{reg}(I(G')), \operatorname{reg}(I(G''))+1\}. \tag{3.1}$$

Note that G' is a unicyclic graph or forest or cycle and also G'' is a unicyclic graph or forest or cycle. Therefore

$$\operatorname{reg}(I(G')) = \nu(G')+1 \leq \nu(G)+1$$

by induction hypothesis or [Zhe04, Theorem 2.18] or [Jac04, Theorem 7.6.28].

By Observation 3.3, $\nu(G'') < \nu(G)$. Then it follows from induction hypothesis or [Zhe04, Theorem 2.18] or [Jac04, Theorem 7.6.28] that

$$\operatorname{reg}(I(G'')) = \nu(G'')+1 \leq \nu(G).$$

Therefore $\operatorname{reg}(I(G)) \leq \nu(G)+1$ by Equation 3.1. □

Lemma 3.9. *(with notation as in Set-up 3.7). If* $\Gamma_3(G) = \emptyset$*, then* $\operatorname{reg}(I(G)) = \nu(G)+1$*, where* $\Gamma_3(G)$ *is the 3-special path set of* G*.*

Proof. By [Kat06, Lemma 2.2], we have $\operatorname{reg}(I(G)) \geq \nu(G)+1$. We use induction on k. If $k=1$, then by Lemma 3.4, $\operatorname{reg}(I(G)) = \nu(G)+1$. Assume that $k > 1$. Let u be a leaf with its unique neighbor v and set $e = \{u, v\}$. Set $G_1 = G \setminus \{v\}$, $G_2 = G \setminus N_G[v]$. Let $\Gamma_3(G_1)$, $\Gamma_3(G_2)$ be the 3-special path sets of G_1 and G_2, respectively. If $\Gamma_3(G_2)$ has an element $Q : z_1 \cdots z_{3m+1}$, for some $m \geq 1$, then there exists an element $P : z_1 \cdots z_{3m_1+1}yvu$ in $\Gamma_3(G)$. This contradicts to $\Gamma_3(G) = \emptyset$. Thus $\Gamma_3(G_2) = \emptyset$. If G_2 is a cycle, there exist $x \in N_{G_2}[v]$ and $x_i \in V(C_n) \cap N_{G_2}(x)$ such that x_i, x, v, u is a path of length 3. Therefore $\Gamma_3(G) \neq \emptyset$. This is contradicts to $\Gamma_3(G) = \emptyset$. Therefore G_2 is either a forest or a unicyclic graph. Then by [Zhe04, Theorem 2.18] or induction hypothesis, we get $\operatorname{reg}(I(G_2)) \leq \nu(G_2)+1$. It follows from Observation 3.3 that $\operatorname{reg}(I(G_2)) \leq \nu(G)$.

Suppose $\Gamma_3(G_1) = \emptyset$. Note that G_1 is either a unicyclic graph, a cycle or a forest. If G_1 is a either unicyclic or forest, then by either induction hypothesis or [Zhe04, Theorem 2.18], $\operatorname{reg}(I(G_1)) \leq \nu(G_1)+1 \leq \nu(G)+1$. If G_1 is a cycle and $n \equiv \{0,1\}(mod\ 3)$, then by [Jac04, Theorem 7.6.28], $\operatorname{reg}(I(G_1)) \leq \nu(G_1)+1 \leq \nu(G)+1$. Suppose G_1 is a cycle and $n \equiv 2(mod\ 3)$. By [Jac04, Theorem 7.6.28], $\operatorname{reg}(I(G_1)) = \nu(C_n)+2$ and it can be easily verified that $\nu(C_n)+1 \leq \nu(G_1)$. Hence it follows from Theorem 2.3 that $\operatorname{reg}(I(G)) \leq \nu(G)+1$.

If $\Gamma_3(G_1) \neq \emptyset$, then there exists an element $Q : z_1z_2 \cdots z_{3m+1}$ in $\Gamma_3(G_1)$, for $m \geq 1$. Then $P' : z_1z_2 \cdots z_{3m+1}vu$ is a path of length $3m+2$ in G. Let $\Gamma_3(G \setminus e)$ and $\Gamma_3(G \setminus N_G[e])$ be the 3-special path sets of $G \setminus e$ and $G \setminus N_G[e]$ respectively. Note that $G \setminus N_G[e] = G_2$. Furthermore, $\Gamma_3(G \setminus e) = \emptyset$ and $\Gamma_3(G \setminus N_G[e]) = \emptyset$. Using induction hypothesis and by Theorem 2.3, we can prove that $\operatorname{reg}(I(G)) \leq \nu(G)+1$. □

For a graph G on n vertices, let $W(G)$ be the *whiskered graph* on $2n$ vertices obtained by adding a pendent vertex (an edge to a new vertex of degree 1) to every vertex of G.

As a consequence of Lemma 3.9, we derive the following result in [MFY17].

Corollary 3.10. [MFY17, Proposition 1.1] *For every integer* $n \geq 3$*, let* C_n *be the* n*-cycle graph and set* $G = W(C_n)$*. Then* $\operatorname{reg}(I(G)) = \nu(G)+1$*.*

Lemma 3.11. *(with notation as in set-up 3.7). If* C_n *is not a connected component of* $G \setminus S$*, then* $\operatorname{reg}(I(G)) = \nu(G)+1$*, where* S *is the 2-special vertex set of* G*.*

Proof. By [Kat06, Lemma 2.2], we have $\text{reg}(I(G)) \geq \nu(G)+1$. We use induction on k. If $k=1$, then by Lemma 3.4, $\text{reg}(I(G)) = \nu(G)+1$. Assume that $k>1$. If $\Gamma_3(G)=\emptyset$, where $\Gamma_3(G)$ is the 3-special path set of G, then by Lemma 3.9, $\text{reg}(I(G)) = \nu(G)+1$. We assume that $\Gamma_3(G)\neq\emptyset$. Let $Q : z_1z_2\cdots z_{3m}z_{3m+1}$ be an element of $\Gamma_3(G)$. Then it follows from Theorem 2.3 that

$$\text{reg}(I(G)) \leq \max\{\text{reg}(I(G\setminus\{z_{3m}\})), \text{reg}(I(G\setminus N_G[z_{3m}]))+1\}. \tag{3.2}$$

Let $G_1 = G\setminus\{z_{3m}\}$, $G_2 = G\setminus N_G[z_{3m}])$ and S', S'' be the 2-special vertex sets of G_1 and G_2 respectively. We can observe that C_n is not a connected component of $G_1\setminus S'$ and $G_2\setminus S''$. Therefore by induction hypothesis and Observation 3.3 we have

$$\text{reg}(I(G_1)) \leq \nu(G_1)+1 \leq \nu(G)+1 \text{ and } \text{reg}(I(G_2)) \leq \nu(G_2)+1 \leq \nu(G).$$

Hence $\text{reg}(I(G)) \leq \nu(G)+1$ by Equation 3.2. □

Lemma 3.12. *Let G be a unicyclic graph with cycle C_n. If $n \equiv 2(\text{mod } 3)$, C_n is a connected component of $G\setminus S$ and $\nu(G\setminus S) < \nu(G)$, then $\text{reg}(I(G)) = \nu(G)+1$, where S is the 2-special vertex set of G.*

Proof. Since C_n is a connected component of $G\setminus S$, $\Gamma_3(G)\neq\emptyset$. Let $Q : z_1z_2\cdots z_{3m}z_{3m+1}$ be an element of $\Gamma_3(G)$. Set $H = G\setminus S = C_n \coprod \coprod_{i=1}^{t} H_i$, where H_i's are trees. Note that $\nu(H) = \nu(C_n)+\nu(H_1)+\ldots+\nu(H_t)$.

We will prove this by induction on t. If $t=1$, then $H = C_n\coprod H_1$ and $m=1$. By Theorem 2.3,

$$\text{reg}(I(G)) \leq \max\{\text{reg}(I(G\setminus z_3)), \text{reg}(I(G\setminus N_G[z_3]))+1\}. \tag{3.3}$$

Let $G\setminus z_3 = G_1\coprod(\coprod_{i=2}^{r} G_i)$, where G_1 is a unicyclic graph and $G_2,\ldots,G_r$ are induced subgraphs of H_1. Let $\Gamma_3(G_1)$ be the 3-special path set of G_1. Note that $\Gamma_3(G_1)=\emptyset$. It follows from Lemma 3.9, $\text{reg}(I(G_1)) = \nu(G_1)+1 = \nu(C_n)+2$. Furthermore, z_4 is an isolated vertex in $G\setminus z_3$. Therefore by [Woo14, Lemma 8],

$$\begin{aligned}\text{reg}(I(G\setminus z_3)) &= \text{reg}(I(G_1)) + \text{reg}(I(\coprod_{i=2}^{r} G_i)) - 1 = \nu(C_n)+2+\nu(\coprod_{i=2}^{r} G_i))\\ &\leq \nu(C_n)+\nu(H_1)+2 \leq \nu(G)+1,\end{aligned}$$

where the last inequality follows from $\nu(H)<\nu(G)$.

Let $G\setminus N_G[z_3] = C_n\coprod(\coprod_{i=2}^{r} G'_i)$, where $G'_2,\ldots,G'_r$ are induced subgraph of H_1. Therefore by [Woo14, Lemma 8],

$$\begin{aligned}\text{reg}(I(G\setminus N_G[z_3])) &= \text{reg}(I(C_n)) + \text{reg}(I(\coprod_{i=2}^{r} G'_i)) - 1 = \nu(C_n)+2+\nu(\coprod_{i=2}^{r} G'_i))\\ &\leq \nu(C_n)+\nu(H_1)+1 \leq \nu(G),\end{aligned}$$

where the last inequality follows from Observation 3.3 and $\nu(H)<\nu(G)$. Hence by Equation 3.3, $\text{reg}(I(G)) \leq \nu(G)+1$. This completes the proof for $t=1$. We assume that $t>1$. By Theorem 2.3,

$$\text{reg}(I(G)) \leq \max\{\text{reg}(I(G\setminus\{z_{3m}\})), \text{reg}(I(G\setminus N_G[z_{3m}]))+1\} \tag{3.4}$$

Let $G \setminus z_{3m} = G_1 \coprod (\coprod_{i=2}^{r} G_i)$, where G_1 is a unicyclic graph and $G_2, \ldots, G_r$ are induced subgraphs of say H_t. Let $\Gamma_3(G_1)$ be the 3-special path set of G_1. If $\Gamma_3(G_1) = \emptyset$, then by Lemma 3.9, $\operatorname{reg}(I(G_1)) = \nu(G_1) + 1$. Therefore by [Woo14, Lemma 8],

$$\operatorname{reg}(I(G \setminus z_{3m}) = \operatorname{reg}(I(G_1)) + \operatorname{reg}(I(\coprod_{i=2}^{r} G_i)) - 1 = \nu(G_1) + 1 + \nu(\coprod_{i=2}^{r} G_i))) \leq \nu(G) + 1.$$

Since G_1 and $\coprod_{i=2}^{r} G_i$ are disjoint induced subgraph of G.

Suppose $\Gamma_3(G_1) \neq \emptyset$. Let S' be the 2-special vertex set of G_1 and $H' = G_1 \setminus S'$. If C_n is not a connected component of H', then by Lemma 3.11, we conclude $\operatorname{reg}(I(G_1)) = \nu(G_1)+1$. By the previous argument, we can prove that $\operatorname{reg}(I(G \setminus \{z_{3m}\})) \leq \nu(G) + 1$.

Suppose C_n is a connected component of H' and $H' = C_n \coprod (\coprod_{i=1}^{r'} H_i')$. If $\nu(H') < \nu(G_1)$, then by induction hypothesis $\operatorname{reg}(I(G_1)) = \nu(G) + 1$. By the previous argument, we can prove that $\operatorname{reg}(I(G \setminus \{z_{3m}\})) \leq \nu(G) + 1$. If $\nu(H') = \nu(G_1)$, then by applying Theorem 3.6, we can write $\operatorname{reg}(I(G_1)) = \nu(G_1) + 2$. Note that H' and H_t are disjoint induced subgraphs of H. Therefore by [Woo14, Lemma 8],

$$\begin{aligned}\operatorname{reg}(I(G \setminus z_{3m}) &= \operatorname{reg}(I(G_1)) + \operatorname{reg}(I(\coprod_{i=2}^{r} G_i)) - 1 = \nu(G_1) + 2 + \nu(\coprod_{i=2}^{r} G_i)) \\ &\leq \nu(H') + 2 + \nu(H_t) \leq \nu(H) + 2 \leq \nu(G) + 1,\end{aligned}$$

where the last inequality follows from $\nu(H) < \nu(G)$.

Let $G \setminus N_G[z_{3m}] = G_1' \coprod (\coprod_{i=2}^{r} G_i')$, where $G_2', \ldots, G_r'$ are induced subgraphs of say $\coprod_{l=1}^{j} H_{i_l}$, where $\{i_1, \ldots, i_j\} \subseteq \{1, \ldots, t\}$. Suppose G_1' is a cycle. Therefore by [Woo14, Lemma 8] and Observation 3.3,

$$\begin{aligned}\operatorname{reg}(I(G \setminus N_G[z_{3m}]) &= \operatorname{reg}(I(G_1')) + \operatorname{reg}(I(\coprod_{i=2}^{r} G_i')) - 1 = \nu(G_1') + 2 + \nu(\coprod_{i=2}^{r} G_i')) \\ &\leq \nu(G_1') + 1 + \nu(\coprod_{l=1}^{j} H_{i_l}) \leq \nu(H) + 1 \leq \nu(G).\end{aligned}$$

Suppose G_1' is a unicyclic graph. Let $\Gamma_3(G_1')$ be the 3-special path set of G_1' and $H' = G_1' \setminus S'$, where S' is the 2-special vertex set of G_1'. If $\Gamma_3(G_1') = \emptyset$, then by Lemma 3.11, we can say $\operatorname{reg}(I(G_1')) = \nu(G_1') + 1$. Therefore by [Woo14, Lemma 8] and Observation 3.3,

$$\begin{aligned}\operatorname{reg}(I(G \setminus N_G[z_{3m}]) &= \operatorname{reg}(I(G_1')) + \operatorname{reg}(I(\coprod_{i=2}^{r} G_i')) - 1 = \nu(G_1') + 1 + \nu(\coprod_{i=2}^{r} G_i')) \\ &\leq \nu(G_1') + \nu(\coprod_{l=1}^{j} H_{i_l}) \leq \nu(G).\end{aligned}$$

Suppose $\Gamma_3(G_1') \neq \emptyset$. If $\nu(H') < \nu(G_1')$, then by induction hypothesis, $\operatorname{reg}(G_1') = \nu(G_1') + 1$. We can prove $\operatorname{reg}(I(G \setminus N_G[z_{3m}]) \leq \nu(G)$ as in the previous argument. If $\nu(H') = \nu(G_1')$, then by Theorem 3.6, $\operatorname{reg}(I(G_1')) = \nu(G_1') + 2$. We can prove $\operatorname{reg}(I(G \setminus N_G[z_{3m}]) \leq \nu(G)$ as in the previous argument. Therefore by Equation 3.4, $\operatorname{reg}(I(G)) \leq \nu(G) + 1$. $\square$

Our one of the main result in this section is now an immediate corollary of Theorem 3.6 and Lemma 3.12.

Corollary 3.13. *Let G be a unicyclic graph with cycle C_n. Then* $\operatorname{reg}(I(G)) = \nu(G) + 2$ *if and only if the following conditions are satisfied:*

1. (1) $n \equiv 2(mod\ 3)$.
2. (2) C_n *is a connected component of* $G \setminus S$*, where* S *is the 2-special vertex set of* G.
3. (3) $\nu(G \setminus S) = \nu(G)$.

Proof. By Theorem 3.2, $\operatorname{reg}(I(G)) = \nu(G) + 1$ or $\operatorname{reg}(I(G)) = \nu(G) + 2$. The proof follows directly from Theorem 3.6 and Lemmas 3.8, 3.11, 3.12. □

Remark 3.14. Let G be a unicyclic graph with cycle C_n. If G satisfies the conditions (1), (2) and (3) from Corollary 3.13, then $\operatorname{reg}(I(G)) > 3$.

Now we are ready to characterize the unicyclic graphs with regularity $\nu(G) + 1$.

Corollary 3.15. *Let G be a unicyclic graph with cycle C_n. Then* $\operatorname{reg}(I(G)) = \nu(G) + 1$ *if and only if*

1. (1) $n \equiv \{0, 1\}(mod\ 3)$*;* *or*
2. (2) C_n *is not a connected component of* $G \setminus S$*, where* S *is the 2-special vertex set of* G*;* *or*
3. (3) $\nu(G \setminus S) < \nu(G)$.

Application of Corollary 3.13 and Corollary 3.15 yields yet another positive result, namely, a partial answer to a question posed by Hà, [Hà14, Problem 6.3].

Corollary 3.16. *Let G be a unicyclic graph with cycle C_n. Then* $\operatorname{reg}(I(G)) = 3$ *if and only if*

1. (1) $\nu(G) = 2$;
2. (2)
   - (a) $n \equiv \{0, 1\}(mod\ 3)$*;* *or*
   - (b) C_n *is not a connected component of* $G \setminus S$*, where* S *is the 2-special vertex set of* G*;* *or*
   - (c) $\nu(G \setminus S) < \nu(G)$.

4. Regularity bounds for colon ideals of paths and cycles

In this section, we study very special colon ideals which are closely related to powers of edge ideals of forests, cycles, and unicyclic graphs. We obtain upper bounds on regularity of these ideals in terms of induced matching number. These bounds are interesting on their own, but they will also be used later in the proofs of our main result. We end this section by providing a linear upper bound for $\operatorname{reg}((I(C_n)^s, f_1, \ldots, f_k))$ when $E(C_n) \cap \{f_1, \ldots, f_k\} = \emptyset$ for $s \geq 2$ (Theorem 4.8).

The following lemma is used repeatedly throughout the paper.

Lemma 4.1. *Let G be a connected graph and let F be a forest with edges $f_1, \ldots, f_k$ such that $E(G) \cap E(F) = \emptyset$. Suppose that every connected component of F, say T, is a rooted tree with root x such that either $V(T) \cap V(G) = \{x\}$ or $V(T) \cap V(G) = \emptyset$. Then we have $s \geq 2$,*

$$\operatorname{reg}((I(G)^s, f_1, \ldots, f_k) : M) = \operatorname{reg}((I(G)^s : M) + I(K_1) + I(K_2)),$$

where M is a minimal generator of $I(G)^{s-1}$ and K_1, K_2 are induced subgraphs of F.

Proof. Since all ideals being discussed are monomial ideals, we then have

$$((I(G)^s, f_1, \dots, f_k) : M) = (I(G)^s : M) + (f_1 : M) + \dots + (f_k : M). \tag{4.1}$$

If there exists $x \in \text{supp}(M)$ such that x divides f_i for some $1 \leq i \leq k$ then it is easy to see that $(f_i : M) =$ (variable). Let $N = \{u_1, \dots, u_p\} \subseteq V(F)$ be the set of all such variables and note that for each u_i there exists a vertex x on G such that $x \in \text{supp}(M)$ and $u_i \in N_F[x]$.

Let $K := F \setminus \{u_1, \dots, u_p\}$ be an induced subgraph of F. Then the edge ideal of K is $I(K) = (g_1, \dots, g_q)$ where $\{g_1, \dots, g_q\} \subseteq \{f_1, \dots, f_k\}$ and none of the elements of N is in $\text{supp}(g_i)$ for any $1 \leq i \leq q$. Note that if there exists a connected component of K with root $x \in V(G)$, by the construction of K we see that $x \notin \text{supp}(M)$. Observe that K has connected components with roots in $V(G)$, denote the induced subgraph containing all these components by K_1, and connected components disjoint from G, denote the induced subgraph containing all these components by K_2. Thus Equation 4.1 becomes:

$$((I(G)^s, f_1, \dots, f_k) : M) = (I(G)^s : M) + I(K_1) + I(K_2) + (u_1, \dots, u_p).$$

Hence we get the desired equality due to [BHT15, Remark 2.6]. □

The following observation is an immediate consequence of the above lemma.

Observation 4.2. Let G, K_1 and K_2 denote the graphs that are introduced in the proof of Lemma 4.1. Since $(V(G) \cup V(K_1)) \cap V(K_2) = \emptyset$ and K_2 is a forest, by [Woo14, Lemma 8] and [Zhe04, Theorem 2.18], we get

$$\begin{aligned} \text{reg}((I(G)^s, f_1, \dots, f_k) : M) &= \text{reg}((I(G)^s : M) + I(K_1)) + \text{reg}(I(K_2)) - 1 \\ &= \text{reg}((I(G)^s : M) + I(K_1)) + \nu(K_2). \end{aligned}$$

The following theorem is needed in the sequel.

Theorem 4.3. *Let G and K_1 denote the graphs that are introduced in the proof of Lemma 4.1. Then for $s \geq 1$,*

$$(I(G)^{s+1} : M) + I(K_1) = (I(G \cup K_1)^{s+1} : M),$$

where M is a minimal generator of $I(G)^s$.

Proof. Let $M = e_1 \cdots e_s$, where $e_1, \dots, e_s \in E(G)$. By the construction of K_1, we see that $V(K_1)$ and $\text{supp}(M)$ does not intersect. Clearly $(I(G)^{s+1} : M) + I(K_1) \subseteq (I(G \cup K_1)^{s+1} : M)$. We need to prove reverse inclusion. By Theorem 2.6, $(I(G \cup K_1)^{s+1} : M)$ is generated by monomials of degree two. Let uv be a minimal generator of $(I(G \cup K_1)^{s+1} : M)$. By Theorem 2.6, either $\{u, v\} \in E(G \cup K_1)$ or u and v are even-connected with respect to M. If $\{u, v\} \in E(G \cup K_1)$, then $uv \in (I(G)^{s+1} : M) + I(K_1)$. Suppose u and v are even-connected in $G \cup K_1$ with respect to M. Then for some $r \geq 1$, let $u = p_0 p_1 \cdots p_{2r} p_{2r+1} = v$ be an even-connection in $G \cup K_1$ where for all $0 \leq l \leq r-1$ and $p_{2l+1} p_{2l+2} = e_i$ for some $1 \leq i \leq s$. Note that $p_i \in \text{supp}(M)$ for all $1 \leq i \leq 2r$. Therefore $p_i \notin V(K_1)$ and $p_i \in V(G)$ for all $1 \leq i \leq 2r$. If $u, v \in V(G)$, then u and v are even-connected in G with respect to M and thus $uv \in (I(G)^{s+1} : M)$. Suppose that $u \in V(K_1) \setminus V(G)$ or $v \in V(K_1) \setminus V(G)$. Then either $p_1 \in V(K_1) \cap V(G)$ or $p_{2r} \in V(K_1) \cap V(G)$, contradiction to $V(K_1) \cap \text{supp}(M) = \emptyset$. Thus $u, v \notin V(K_1)$. Therefore u and v are even-connected in G with respect to M. Hence $uv \in (I(G)^{s+1} : M)$. □

The following result assists the proof of Lemma 4.7 and Theorem 4.8.

Lemma 4.4. *Let P_n be a path of length n and F_k be a forest with edges $f_1, \ldots, f_k$ such that $E(P_n) \cap E(F_k) = \emptyset$. Suppose that every connected component of F_k, say T, is a rooted tree with root x such that either $V(T) \cap V(P_n) = \{x\}$ or $V(T) \cap V(P_n) = \emptyset$. Let $G = P_n \cup F_k$. Then for $s \geq 1$,*

$$\operatorname{reg}(((I(P_n)^{s+1}, f_1, \ldots, f_k) : M)) \leq \nu(G) + 1,$$

where M is a minimal generator of $I(P_n)^s$.

Proof. If $F_k = \emptyset$, then by [JNS16, Corollary 4.12] and [Zhe04, Theorem 2.18] we have

$$\operatorname{reg}(I(P_n)^{s+1} : M) \leq \operatorname{reg}(I(P_n)) = \nu(G) + 1.$$

We use the same set up as in the proof of Lemma 4.1 and Observation 4.2. Thus it suffices to show that

$$\operatorname{reg}((I(P_n)^{s+1} : M) + I(K_1)) + \nu(K_2) \leq \nu(G) + 1.$$

Let H be the graph associated to $I(P_n \cup K_1)$. Since H is a forest, by [JNS16, Corollary 4.12] and [Zhe04, Theorem 2.18],

$$\operatorname{reg}(I(H)^{s+1} : M) \leq \operatorname{reg}(I(H)) = \nu(H) + 1.$$

Since H and K_2 are disjoint induced subgraph of G, we have $\nu(H)+\nu(K_2) \leq \nu(G)$. Therefore,

$$\operatorname{reg}((I(P_n)^{s+1} : M) + I(K_1)) + \nu(K_2) \leq \nu(H) + \nu(K_2) + 1 \leq \nu(G) + 1.$$

□

Set-up 4.5. Let G be a unicyclic graph with cycle C_n and $I_0 = I(C_n)$. Let F be the graph with $E(F) = E(G) \setminus E(C_n) = \{f_1, \ldots, f_k\}$. Let M be a minimal generator of I_0^s for some $s \geq 1$. Then by Theorem 2.6,

$$(I_0^{s+1} : M) = I_0 + I_1 + I_2,$$

where $I_1 = (\{uv \mid u \text{ is even-connected to } v \text{ in } C_n \text{ with respect to } M \text{ and } u \neq v\})$ and $I_2 = (\{u^2 \mid u \text{ is even-connected to itself in } C_n \text{ with respect to } M\})$. We use the same set up as in the proof of Lemma 4.1. Therefore

$$((I_0^{s+1}, f_1, \ldots, f_k) : M) = (I_0^{s+1} : M) + I(K_1) + I(K_2) = I_0 + I_1 + I_2 + I(K_1) + I(K_2).$$

Let $\mathcal{C}$, $\mathcal{C}'$ and H be the graph associated to $(\widetilde{I_0^{s+1} : M})$, $I_0 + I_1$ and $I_0 + I(K_1)$ respectively. By [BHT15, Proof of Theorem 5.2], $\operatorname{reg}(I(\mathcal{C})) \leq \nu(C_n) + 1$. Since $\mathcal{C}'$ is an induced subgraph of $\mathcal{C}$, by Theorem 2.3,

$$\operatorname{reg}(I(\mathcal{C}')) \leq \nu(C_n) + 1. \tag{4.2}$$

Lemma 4.6. *(with notation as in Set-up 4.5). Then*

$$\operatorname{reg}(I(\mathcal{C}') + I(K_1)) \leq \nu(H) + 1$$

Proof. If $E(K_1) = \emptyset$, then by Equation 4.2, $\operatorname{reg}(I(\mathcal{C}')) \leq \nu(H)+1$. Let $E(K_1) = \{g_1, \ldots, g_l\}$, where $\{g_1, \ldots, g_l\} \subseteq \{f_1, \ldots, f_k\}$ and G' be the graph associated to $I(\mathcal{C}') + (g_1, \ldots, g_l)$. We use induction on l. If $l = 1$, then there is a leaf y in H such that $g_1 = \{x_i, y\}$ for some $1 \leq i \leq n$. Note that y is still a leaf in G' by Theorem 4.3. It follows from [BC15, Lemma 3.25] that

$$\operatorname{reg}(I(G')) = \max\{\operatorname{reg}(I(\mathcal{C}')), \operatorname{reg}(I(\mathcal{C}' \setminus N_{\mathcal{C}'}[x_i])) + 1\}. \tag{4.3}$$

By Equation 4.2, $\operatorname{reg}(I(\mathcal{C}')) \leq \nu(C_n)+1 \leq \nu(H)+1$. We may assume $x_i = x_1$. Let P be a path with $V(P) = \{x_3, \ldots, x_{n-1}\} \subset V(C_n)$. By [ABar, Proposition 3.5], $(I(P)^{s+1} : M)$ is a quadratic squarefree monomial ideal. Let P' be the graph associated to $(I(P)^{s+1} : M)$. Therefore

$$\begin{aligned} \operatorname{reg}(I(P')) &\leq \operatorname{reg}(I(P)) && \text{(by [JNS16, Corollary 4.12 (2)])} \\ &= \nu(P)+1 && \text{(by [Zhe04, Theorem 2.18])} \\ &\leq \nu(H). && \text{(by Observation 3.3)} \end{aligned}$$

Observe that $\mathcal{C}' \setminus N_{\mathcal{C}'}[x_i]$ is an induced subgraph of P'. Thus $\operatorname{reg}(I(\mathcal{C}' \setminus N_{\mathcal{C}'}[x_i])) \leq \nu(H)$ by Theorem 2.3. Therefore $\operatorname{reg}(I(G')) \leq \nu(H)+1$ by Equation 4.3 and this completes the proof for $l = 1$.

Assume that $l > 1$. Then there exists a leaf y in G' such that $\{x\} = N_H(y)$ and $g_i = \{x, y\}$ for some $1 \leq i \leq l$. It follows from [BC15, Lemma 3.25] that

$$\operatorname{reg}(I(G')) = \max\{\operatorname{reg}(I(G' \setminus g_i)), \operatorname{reg}(I(G' \setminus N_{G'}[x])) + 1\}. \tag{4.4}$$

We can see that $I(G' \setminus g_i) = I(\mathcal{C}') + (g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_l)$. By induction hypothesis we have

$$\operatorname{reg}(I(G' \setminus g_i)) \leq \nu(H \setminus g_i) + 1 \leq \nu(H) + 1.$$

The lemma is complete after obtaining $\operatorname{reg}(I(G' \setminus N_{G'}[x]) \leq \nu(H)$. Since M is a minimal generator of I_0^s, $M = e_1 \cdots e_s$ where $\{e_1, \ldots, e_s\} \subseteq E(C_n)$ and e_i's are not necessarily distinct. Set $\{e_{i_1}, \ldots, e_{i_t}\} = \{e_1, \ldots, e_s\} \cap E(G' \setminus N_{G'}[x])$. We need to analyze three cases.

Case 1: Suppose $N_{G'}[x] \cap V(C_n) = \emptyset$. Then by Theorem 4.3,

$$I(G' \setminus N_{G'}[x])) = I(\mathcal{C}') + (g_{i_1}, \ldots, g_{i_r}),$$

where $\{g_{i_1}, \ldots, g_{i_r}\} = \{g_1, \ldots, g_l\} \cap E(G' \setminus N_{G'}[x]))$. By induction hypothesis and Observation 3.3,

$$\operatorname{reg}(I(G' \setminus N_{G'}[x])) \leq \nu(H \setminus N_H[x]) + 1 \leq \nu(H).$$

If $I(G' \setminus N_{G'}[x])) = I(\mathcal{C}')$, then by using Equation 4.2 and Observation 3.3, we can write $I(G' \setminus N_{G'}[x])) \leq \nu(H \setminus N_H[x]) + 1 \leq \nu(H)$.

Case 2: Suppose $x_i \in N_{G'}[x]$ and $x_i \neq x$, where $x_i \in V(C_n)$, for some $1 \leq i \leq n$. Without loss of generality we assume that $x_i = x_1$. Let P be the path on the vertices $\{x_2, \ldots, x_n\}$. Let $\{g_{i_1}, \ldots, g_{i_r}\} = \{g_1, \ldots, g_l\} \cap E(G' \setminus N_{G'}[x])$. It can easily be verified that

$$I(G' \setminus N_{G'}[x]) = (I(P)^{t+1} : e_{i_1} \cdots e_{i_t}) + (g_{i_1}, \ldots, g_{i_r}).$$

Furthermore, by Theorem 4.3,

$$((I(P), g_{i_1}, \ldots, g_{i_r})^{t+1} : e_{i_1} \cdots e_{i_t}) = (I(P)^{t+1} : e_{i_1} \cdots e_{i_t}) + (g_{i_1}, \ldots, g_{i_r}).$$

Let H' be the graph associated to $I(P) + (g_{i_1}, \ldots, g_{i_r})$. Then by Lemma 4.4, we conclude $\operatorname{reg}(I(G' \setminus N_{G'}[x])) \leq \nu(H') + 1$. Therefore by Observation 3.3,

$$\operatorname{reg}(I(G' \setminus N_{G'}[x])) \leq \nu(H).$$

Case 3: Suppose $x = x_i$, for some $1 \leq i \leq n$. Without loss of generality we assume that $x_i = x_1$. Let P be the path on the vertices $\{x_3, \ldots, x_{n-1}\}$. Set $\{g_{j_1}, \ldots, g_{j_r}\} = \{g_1, \ldots, g_l\} \cap E(G' \setminus N_{G'}[x])$. By Theorem 4.3,

$$(I(P), g_{j_1}, \ldots, g_{j_r})^{t+1} : e_{i_1} \cdots e_{i_t}) = (I(P)^{t+1} : e_{i_1} \cdots e_{i_t}) + (g_{j_1}, \ldots, g_{j_r}).$$

Let H' be the graph associated to $I(P)+(g_{j_1},\dots,g_{j_r})$. Then by Lemma 4.4 and Observation 3.3, we conclude that

$$\operatorname{reg}(I(P),g_{j_1},\dots,g_{j_r})^{t+1} : e_{i_1}\cdots e_{i_t}) \leq \nu(H')+1 \leq \nu(H).$$

Let P' be the graph associated to $(I(P)^{t+1} : e_{i_1}\cdots e_{i_t})+(g_{j_1},\dots,g_{j_r})$. Then it can easily be verified that $G'\setminus N_{G'}[x]$ is an induced subgraph of P'. Therefore $\operatorname{reg}(I(G'\setminus N_{G'}[x])) \leq \nu(H)$ by Theorem 2.3 and the lemma is proved. $\square$

The following lemma will be used in the proof of Theorem 4.8.

Lemma 4.7. *(with notation as in Set-up 4.5). Then for $s \geq 1$,*

$$\operatorname{reg}((I_0^{s+1}, f_1,\dots,f_k) : M) \leq \nu(G)+1,$$

Proof. Let $I_0 = (x_1x_2,\dots,x_nx_1)$. By Observation 4.2, we need to prove that,

$$\operatorname{reg}(I(\mathcal{C})+I(K_1))+\nu(K_2) \leq \nu(G)+1, \tag{4.5}$$

where $\mathcal{C}$ is the graph associated to $(\widetilde{I_0^{s+1} : M})$. Suppose $I_2 = (x_{i_1}^2,\dots,x_{i_m}^2)$. Then

$$I(\mathcal{C}) = I_0+I_1+(x_{i_1}z_1,\dots,x_{i_m}z_m),$$

where $z_1,\dots,z_m$ are new vertices. Let L be the graph associated to $I(\mathcal{C})+I(K_1)$. By [BC15, Lemma 3.25], either $\operatorname{reg}(I(L)) = \operatorname{reg}(I(L\setminus z_1))$ or $\operatorname{reg}(I(L)) = \operatorname{reg}(I(L\setminus N_L[x_{i_1}]))+1$.

Suppose $\operatorname{reg}(I(L)) = \operatorname{reg}(I(L\setminus N_L[x_{i_1}]))+1$. We see that $\{x_1,\dots,x_n,z_1\} \subseteq N_L[x_{i_1}]$ by [BHT15, Proof of Theorem 5.2]. Therefore

$$I(L\setminus N_L[x_{i_1}])) = (z_2,\dots,z_m)+I(K_1\setminus\{x_{j_1},\dots,x_{j_r}\}),$$

where $\{x_{j_1},\dots,x_{j_r}\} = \{x_1,\dots,x_n\}\cap V(K_1)$. Since M is a minimal generator of $I(C_n)^s$, there exists an edge $\{x_p,x_{p+1}\}$ such that x_px_{p+1} divides M. This means neither x_p nor x_{p+1} is a root of a tree in K_1. Let Ω be an induced matching in $K_1\setminus\{x_{j_1},\dots,x_{j_r}\}$. Then $\Omega\cup\{x_p,x_{p+1}\}$ is an induced matching in L. Therefore,

$$\operatorname{reg}(I(L)) = \operatorname{reg}(I(L\setminus N_L[x_{i_1}]))+1 \leq \nu(L\setminus N_L[x_{i_1}]))+2 \leq \nu(L)+1.$$

Since L and K_2 are disjoint induced subgraphs of G, we have $\nu(L)+\nu(K_2) \leq \nu(G)$. From Equation 4.5, we have

$$\operatorname{reg}(I(\mathcal{C})+I(K_1))+\nu(K_2) \leq \nu(L)+\nu(K_2)+1 \leq \nu(G)+1.$$

Therefore we proved Equation 4.5.

Suppose $\operatorname{reg}(I(L)) = \operatorname{reg}(I(L\setminus z_1))$. Let $L_i = L\setminus\{z_1,\dots,z_i\}$, for $1 \leq i \leq m$. By [BC15, Lemma 3.25], either $\operatorname{reg}(I(L_1)) = \operatorname{reg}(I(L_2))$ or $\operatorname{reg}(I(L_1)) = \operatorname{reg}(I(L_1\setminus N_{L_1}[x_{i_2}]))+1$. If $\operatorname{reg}(I(L_1)) = \operatorname{reg}(I(L_1\setminus N_{L_1}[x_{i_2}]))+1$, then by the previous argument, we can prove the Equation 4.5. If $\operatorname{reg}(I(L_1)) = \operatorname{reg}(I(L_2))$, then one proceeds in the same manner. If at any stage $\operatorname{reg}(I(L_j)) = \operatorname{reg}(I(L_j\setminus N_{L_j}[x_{i_{j+1}}]))+1$, then we are done. If not, we get

$$\operatorname{reg}(I(L)) = \operatorname{reg}(I(L_1)) = \cdots = \operatorname{reg}(I(L_m)).$$

Therefore now we need to prove that,

$$\operatorname{reg}(I(\mathcal{C}')+I(K_1))+\nu(K_2) \leq \nu(G)+1, \tag{4.6}$$

where $\mathcal{C}'$ is the graph associated to $I_0 + I_1$. Since H and K_2 are disjoint induced subgraphs of G, we have $\nu(H) + \nu(K_2) \leq \nu(G)$. Therefore by Lemma 4.6 and Equation 4.6,

$$\operatorname{reg}(I(\mathcal{C}') + I(K_1)) + \nu(K_2) \leq \nu(H) + 1 + \nu(K_2) \leq \nu(G) + 1$$

Suppose $I_2 = 0$. Then we have $((I_0^{s+1}, f_1, \ldots, f_k) : M) = I_0 + I_1 + I(K_1) + I(K_2)$. Let $\mathcal{C}'$ and H be the graph associated to $I_0 + I_1$ and $I_0 + I(K_1)$ respectively. By Lemma 4.6,

$$\operatorname{reg}(I(\mathcal{C}') + I(K_1)) \leq \nu(H) + 1.$$

As in the previous case, we can prove that $\operatorname{reg}((I_0^{s+1}, f_1, \ldots, f_k) : M) \leq \nu(G) + 1$. □

We are now ready to prove the main result of this section.

Theorem 4.8. *(with notation as in Set-up 4.5). Then for $s \geq 2$,*

$$\operatorname{reg}((I_0^s, f_1, \ldots, f_k)) \leq 2s + \nu(G) - 1,$$

Proof. Let $s \geq 2$ and $\{m_1, \ldots, m_q\}$ be the minimal generators of I_0^{s-1} also let $J = (I_0^s, f_1, \ldots, f_k)$. Then consider the following short exact sequence

$$0 \longrightarrow \frac{R}{(J : m_1)}(-2) \longrightarrow \frac{R}{J} \longrightarrow \frac{R}{(J, m_1)} \longrightarrow 0. \tag{4.7}$$

Furthermore, let $J_l = (J, m_1, \ldots, m_l)$ for $1 \leq l \leq q$ and set $J_0 = J$. Then, for $0 \leq l \leq q-1$, we have

$$0 \longrightarrow \frac{R}{(J_l : m_{l+1})}(-2) \longrightarrow \frac{R}{J_l} \longrightarrow \frac{R}{(J_{l+1})} \longrightarrow 0. \tag{4.8}$$

Putting Equation 4.7 and Equation 4.8 together, we get

$$\operatorname{reg}(J) \leq \max\{\operatorname{reg}(J_l : m_{l+1}) + 2, 0 \leq l \leq q-1, \operatorname{reg}(I_0^{s-1}, f_1, \ldots, f_k)\}. \tag{4.9}$$

It follows from [Ban15, Theorem 4.12], that

$$(J_l : m_{l+1}) = (I_0^s : m_{l+1}) + (f_1 : m_{l+1}) + \ldots + (f_k : m_{l+1}) + (\text{variables}).$$

Then by using [KM06, Theorem 1.4], Lemma 4.7 and Theorem 4.3 we have

$$\begin{aligned} \operatorname{reg}\ (J_l : m_{l+1}) &\leq \operatorname{reg}((I_0^s : m_{l+1}) + (f_1 : m_{l+1}) + \ldots + (f_k : m_{l+1})) \\ &= \operatorname{reg}((I_0^s, f_1, \ldots, f_k) : m_{l+1}) \leq \nu(G) + 1. \end{aligned} \tag{4.10}$$

We use induction on s. Suppose $s = 2$. From the Equation 4.9 for $s = 2$ we have

$$\operatorname{reg}(J) \leq \max\{\operatorname{reg}\ (J_l : m_{l+1}) + 2, 0 \leq l \leq q-1, \operatorname{reg}(I_0, f_1, \ldots, f_k)\}.$$

Since from Theorem 3.2, we know that $\operatorname{reg}(I_0, f_1, \ldots, f_k) \leq \nu(G) + 2$, from Equation 4.10 we have $\operatorname{reg}(J) \leq \nu(G) + 3$.

Let $s > 2$. From induction hypothesis we have

$$\operatorname{reg}\ (I_0^{s-1}, f_1, \ldots, f_k) \leq 2(s-1) + \nu(G) - 1.$$

So induction hypothesis with Equation 4.9 and Equation 4.10 settle our assertion. □

5. Regularity of powers of unicyclic graphs

In this section, we obtain precise expressions for the regularity of powers of edge ideals of unicyclic graphs. To prove this, we shall first establish the upper bound for $\operatorname{reg}(I(G)^s)$, for all $s \geq 1$.

We begin with the following observation.

Observation 5.1. Let G be a unicyclic graph. Then there exist a cycle C_n and a forest F such that $E(G) = E(C_n) \cup E(F)$ and $E(C_n) \cap E(F) = \emptyset$. Note that $n \geq 3$, and $k \geq 1$. Let f_1 be a leaf of G. Then $I(G) = I(G \setminus f_1) + (f_1)$ and note that $G_1 := G \setminus f_1$ is an induced subgraph of G. If $k \geq 2$, then let f_2 be a leaf of G_1. Similarly, $I(G_1) = I(G_1 \setminus f_2) + (f_2)$ and $G_2 := G_1 \setminus f_2$ is an induced subgraph of G_1. Let $G_0 = G$. We set up G_i to be the graph obtained from G_{i-1} by deleting the f_i where f_i is a leaf of G_{i-1} for $1 \leq i \leq k$. Then $G_k = C_n$. By following this fashion, we conclude that

$$I(G) = I(G_1) + (f_1) = \ldots = I(C_n) + (f_1, \ldots, f_k)$$

Since f_1 is a leaf in G, we have the following identity:

$$I(G)^s = I(G_1)^s + \sum_{j=1}^{s} I(G_1)^{s-j} f_1^j$$

Therefore, we get the following equalities:

$$\begin{aligned}
(I(G)^s, f_1) &= (I(G_1)^s, f_1) \\
&\vdots \\
(I(G)^s, f_1, \ldots, f_i) &= (I(G_i)^s, f_1, \ldots, f_i) \\
&\vdots \\
(I(G)^s, f_1, \ldots, f_k) &= (I(G_k)^s, f_1, \ldots, f_k) = (I(C_n)^s, f_1, \ldots, f_k)
\end{aligned}$$

We now prove an upper bound for $\operatorname{reg}(I(G)^s)$, when G is a unicyclic graph.

Lemma 5.2. *If G is a unicyclic graph, then for all $s \geq 1$,*

$$\operatorname{reg}(I(G)^s) \leq 2s + \operatorname{reg}(I(G)) - 2.$$

Proof. The statement is clear for $s = 1$. So, assume that $s \geq 2$. Let C_n be the cycle that is contained in G with its edge ideal $I_0 = I(C_n)$ and let $\{m_1, \ldots, m_q\}$ be the minimal generators of I_0^{s-1}. Let F be the graph with $E(F) = E(G) \setminus E(C_n)$ and $I(F) = (f_1, \ldots, f_k)$. Let $G_i = G \setminus \{f_1, \ldots, f_i\}$ and f_{i+1} is a leaf in G_i for $0 \leq i \leq k-1$, $G_k = C_n$. We consider the following short exact sequence:

$$0 \longrightarrow \frac{R}{(I(G)^s : f_1)}(-2) \longrightarrow \frac{R}{I(G)^s} \longrightarrow \frac{R}{(I(G)^s, f_1)} \longrightarrow 0.$$

Then by Observation 5.1 and the short exact sequence yield to

$$\operatorname{reg}(I(G)^s) \leq \max\{\operatorname{reg}(I(G)^s : f_1) + 2, \operatorname{reg}(I(G_1)^s, f_1)\}.$$

Since f_1 is a leaf of G, by [Mor10, Lemma 2.10], $\operatorname{reg}(I(G)^s : f_1) = I(G)^{s-1}$. Then by induction hypothesis, we have $\operatorname{reg}(I(G))^{s-1} + 2 \leq 2s + \operatorname{reg}(I(G)) - 2$. We need to show that $\operatorname{reg}(I(G_1)^s, f_1)) \leq 2s + \operatorname{reg}(I(G)) - 2$.

Claim: $\operatorname{reg}((I(G_i)^s, f_1, \ldots, f_i)) \leq 2s + \operatorname{reg}(I(G)) - 2$ for all $1 \leq i \leq k$ and $s \geq 1$.
Proof of claim: We use induction on s. If $s = 1$, then $\operatorname{reg}((I(G_i), f_1, \ldots, f_i)) = \operatorname{reg}(I(G))$. Assume that $s \geq 2$. If $i = k$, then by Theorem 4.8,

$$\operatorname{reg}(I(G_k)^s, f_1, \ldots, f_k) \leq 2s + \nu(G) - 1 \leq 2s + \operatorname{reg}(I(G)) - 2.$$

We need to prove that $\operatorname{reg}(I(G_i)^s, f_1, \ldots, f_i) \leq 2s + \operatorname{reg}(I(G)) - 2$ for all $1 \leq i \leq k-1$. Let $J_i = (I(G_i)^s, f_1, \ldots, f_i)$ for $1 \leq i \leq k-1$. Then we have the following short exact sequence

$$0 \longrightarrow \frac{R}{(J_i : f_{i+1})}(-2) \longrightarrow \frac{R}{J_i} \longrightarrow \frac{R}{(J_i, f_{i+1})} \longrightarrow 0. \tag{5.1}$$

We have,

$$\operatorname{reg}(J_i) \leq \max\{\operatorname{reg}(J_l : f_{l+1}) + 2, i \leq l \leq k-1,\ \operatorname{reg}(I_0^s, f_1, \ldots, f_k)\}.$$

By Theorem 4.8,

$$\operatorname{reg}(I_0^s, f_1, \ldots, f_k) \leq 2s + \nu(G) - 1 \leq 2s + \operatorname{reg}(I(G)) - 2.$$

We would like to show that $\operatorname{reg}(J_l : f_{l+1}) \leq 2(s-1) + \operatorname{reg}(I(G)) - 2$. Let L be the graph with the edges $f \in \{f_1, \ldots, f_l\}$ such that $f \cap N_G[f_{l+1}] = \emptyset$ and if there exists $g \in \{f_1, \ldots, f_l\}$ such that $g \cap N_G[f_{l+1}] \neq \emptyset$, then $f \cap g = \emptyset$. Clearly, L is an induced subgraph of F. Note that if $f \cap N_G[f_{l+1}] = \emptyset$, then $(f : f_{l+1}) = (f)$ and if $g \cap N_G[f_{l+1}] \neq \emptyset$ then $(g : f_{l+1}) = (\text{variable})$. By Observation 5.1, we observe that

$$\begin{aligned}(J_l : f_{l+1}) &= (I(G_l)^s : f_{l+1}) + (f_1 : f_{l+1}) + \ldots + (f_l : f_{l+1}) \\ &= I(G_l)^{s-1} + I(L) + (\text{variables}).\end{aligned}$$

Therefore by [KM06, Theorem 1.4],

$$\operatorname{reg}((J_l : f_{l+1})) \leq \operatorname{reg}(I(G_l)^{s-1} + I(L)). \tag{5.2}$$

If $E(L) = \emptyset$, then by induction hypothesis applied to the Theorem 5.2 yield to

$$\operatorname{reg}((J_l : f_{l+1})) \leq \operatorname{reg}(I(G_l)^{s-1}) \leq 2(s-1) + \operatorname{reg}(I(G_l)) - 2.$$

Thus the claim holds since G_l is an induced subgraph of G.

Suppose that $E(L) \neq \emptyset$. Let K_1 be the induced subgraph of G with edges in $E(L)$ such that $e \in E(F)$ is an edge of a rooted tree with root in $V(C_n)$. Let K_2 be the induced subgraph L such that $E(K_2) = E(L) \setminus E(K_1)$. Finally, let H be the graph with the edge set $E(H) = E(G_l) \cup E(K_1)$. It is easy to see from the definition of L and K_2 that K_2 and H are disjoint. Then Equation 5.2 becomes

$$\operatorname{reg}((J_l : f_{l+1})) \leq \operatorname{reg}(I(G_l)^{s-1} + I(K_1)) + \nu(K_2)$$

Applying induction hypothesis to H, K_1 with power $(s-1)$, we have

$$\operatorname{reg}(I(G_l)^{s-1} + I(K_1)) \leq 2(s-1) + \nu(H) - 1. \tag{5.3}$$

Putting Equations (5.2 and 5.3) together we get

$$\operatorname{reg}((J_l : f_{l+1})) \leq 2(s-1) + \nu(H) + \nu(K_2) - 1$$

Since K_2 and H are disjoint induced subgraphs of G, we have $\nu(H) + \nu(K_2) \leq \nu(G)$. Hence the claim is proved. Therefore the theorem is proved. ∎

Now, we present the main result of this paper.

Theorem 5.3. *If G is a unicyclic graph, then for all $s \geq 1$,*

$$\operatorname{reg}(I(G)^s) = 2s + \operatorname{reg}(I(G)) - 2.$$

Proof. Let G be unicyclic graph. Suppose $\operatorname{reg}(I(G)) = \nu(G) + 1$. Then by Lemma 5.2 and [BHT15, Theorem 4.5], for all $s \geq 1$

$$\operatorname{reg}(I(G)^s) = 2s + \nu(G) - 1 = 2s + \operatorname{reg}(I(G)) - 2.$$

Suppose $\operatorname{reg}(I(G)) = \nu(G) + 2$. By Corollary 3.13,

$$H = G \setminus S = C_n \coprod (\coprod_{i=1}^{t} H_i),$$

where S is the 2-special vertex set of G. Note that $n \equiv 2 (mod\ 3)$.

Claim: $\operatorname{reg}(I(H)^s) = 2s + \nu(H)$, for all $s \geq 1$.

Proof of claim: We prove the result using induction on t. Let $t = 1$. Let $H = C_n \coprod H_1$. By [HTT16, Lemma 2.5], we can prove the case $s = 1$. Then by [HTT16, Proposition 2.7], we get

$$\operatorname{reg}(I(H)^2) = \operatorname{reg}(I(C_n)) + \operatorname{reg}(I(H_1)^2) - 1 = \nu(C_n) + 2 + 3 + \nu(H_1) - 1 = 4 + \nu(H).$$

By [NV16, Theorem 5.7], for $s \geq 3$, we have

$$\operatorname{reg}(I(H)^s) = 2s + \nu(H).$$

This completes the proof for $t = 1$. Suppose $t \geq 2$. Let $H' = C_n \coprod (\coprod_{i=1}^{t-1} H_i)$. Then $H = H' \coprod H_t$ and $\nu(H) = \nu(H') + \nu(H_t)$. By using induction hypothesis for H' we have

$$\operatorname{reg}(I(H'^s)) = 2s + \nu(H') \quad \text{for } s \geq 1.$$

Since H_t is a tree, by [BHT15, Theorem 4.7] we have

$$\operatorname{reg}(I(H_t)^s) = 2s + \nu(H_t) - 1 \quad \text{for } s \geq 1.$$

By [NV16, Theorem 5.7], for $s \geq 2$, we have

$$\operatorname{reg}(I(H)^s) = 2s + \nu(H).$$

Hence the claim.

By Corollary 3.13, $\nu(H) = \nu(G)$. Hence $\operatorname{reg}(I(H)^s) = 2s + \nu(G)$. Therefore it follows from [BHT15, Corollary 4.3] and Lemma 5.2 that for all $s \geq 1$,

$$\operatorname{reg}(I(G)^s) = 2s + \nu(G) = 2s + \operatorname{reg}(I(G)) - 2.$$

Hence the theorem is proved. □

Since whiskered cycle graphs are unicyclic graphs, we derive, from Corollary 3.10 and Theorem 5.3, main results of Moghimian, Fakhari and Yassemi.

Corollary 5.4. [MFY17, Theorem 2.5] *Let $G = W(C_n)$ be a whiskered cycle graph. Then for all $s \geq 1$,*

$$\operatorname{reg}(I(G)^s) = 2s + \nu(G) - 1.$$

We conclude the paper with the following remark.

Remark 5.5. So far, we have been talking about regularity powers of connected unicyclic graphs. Now we are ready to give a precise expression for $\operatorname{reg}(I(G)^s)$ when G is a non-connected unicyclic graphs. Suppose $G = G_1 \coprod(\coprod_{i=2}^t G_i)$, where G_1 is a connected unicyclic graph and $G_2, \ldots, G_t$ are trees. By Theorem 5.3 and [BHT15, Theorem 4.7], we have

(1) $\operatorname{reg}(I(G_1)^s) = 2s + \operatorname{reg}(I(G_1)) - 2$, for all $s \geq 1$.
(2) $\operatorname{reg}(I(\coprod_{i=2}^t G_i)^s) = 2s + \nu(\coprod_{i=2}^t G_i) - 1 = 2s + \operatorname{reg}(I(\coprod_{i=2}^t G_i)) - 2$, for all $s \geq 1$.

By [NV16, Theorem 5.7], $\operatorname{reg}(I(G)^s) = 2s + \operatorname{reg}(I(G)) - 2$, for all $s \geq 2$.

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